GEOMETRY OF MAGNETOHYDRODYNAMIC FLOW

G. PURUSHOTHAM

Govt. Nagarjunasagar Engineering College, Hyderabad

&

MADHUSUDAN

Agarwal Evening Science College, Hyderabad

(*Received* 4 *August* 1970)

The geometry of streamline and magnetic lines have been considered to study the various kinematic and kinetic properties of magnetohydrodynamic flows.

Wasserman[1] has examined the geometry of streamline and fieldline congruences, in order to study the various kinematic properties of magnetohydrodynamic flows. He has also precisely shown, to what extent the geometry of streamlines and fieldlines determine MHD flow. Though his approach has yielded some new interesting results, but description of geometry seems to be circutous. Consequently herein we have attempted to describe the geometry of streamline and fieldline in a more direct way and applied these to study the kinematic properties of magnetohydrodynamic flow. Effecting the transformation in intrinsic form the momentum equations, it is observed that the equation derived by Wasserman[1] and non-magnetic flow can be deduced as a special case from this investigation. Kinetic energy potential is introduced which depends upon the geometry of streamline only. It is proved that the momentum per unit mass is conserved along an individual streamline. Field relations are decomposed in streamline geometry. Irrotational character of the electric field is expressed in intrinsic form. Introducing the velocity of sound momentum relations are decomposed.

BASIC EQUATIONS

The basic equations governing steady flow of magnetogasdynamics, in the absence of viscosity, thermal conductivity and electrical resistance are given below in the usual notation[2]

$$\text{div}\,(\rho \vec{q}) = 0 \qquad (1)$$

$$\rho\,(\vec{q}\cdot\nabla)\vec{q} + \nabla\left(p + \frac{\omega^2}{2}\right) = (\vec{\omega}\cdot\nabla)\vec{\omega} \qquad (2)$$

$$\vec{q}\cdot\nabla S = 0 \qquad (3)$$

$$\text{curl}\,(\vec{\omega}\wedge\vec{q}) = 0 \qquad (4)$$

$$\text{div}\,\vec{\omega} = 0 \qquad (5)$$

$$\vec{E} + (\mu_e)^{\frac{1}{2}}\,\vec{q}\wedge\vec{\omega} = 0 \qquad (6)$$

$$\rho = p^{1/\gamma}\, e^{-S/JC_v} \qquad (7)$$

where $\vec{q}$, ρ, p, S and $(\vec{\omega} = (\mu_e)^{\frac{1}{2}}\vec{H})$ are the velocity vector field, the fluid density, the hydrodynamic pressure, the specific entropy and the magnetic field, and JC_v, C_v, γ are the Joule's constant, the specific heat at constant volume and the ratio of specific heats at constant pressure and constant volume respectively.

GEOMETRIC RESULTS

Considering $\vec{s}$, $\vec{n}$ and $\vec{b}$ as the unit tangent triply orthogonal vectors to the curves of congruences formed by the streamlines, their principal normals and binormals respectively and denoting $\frac{\delta}{\delta s}$, $\frac{\delta}{\delta n}$, $\frac{\delta}{\delta b}$ as intrinsic derivatives along these curves and also selecting, $\vec{\mu}$ as the position vector on a streamline we have the following results[3]:

$$\frac{\delta\vec{\mu}}{\delta s} = \vec{s} = \frac{\vec{q}}{|\vec{q}|} \qquad (8)$$

$$\frac{\delta \vec{s}}{\delta s} = \vec{n}\, K_s \qquad (9) \qquad\qquad \frac{\delta \vec{n}}{\delta s} = \vec{b}\, \tau_s - \vec{s}\, K_s \qquad (11)$$

$$\frac{\delta \vec{b}}{\delta s} = -\tau_s\, \vec{n} \qquad (10) \qquad\qquad \vec{D} = \tau_s\, \vec{s} + K_s\, \vec{b} \qquad (12)$$

where K_s, τ_s and $\vec{D}$ are the curvature, the torsion and the Darboux vector for the curve having unit tangent vector $\vec{S}$ (streamline).

In addition to these, we take primary parameters of the streamline ($\vec{S}$) geometry, the four variables θ_{ns}, θ_{bs}, ϕ ψ_s and Ω_s defined by Bjϕrgum[4] expression for $\nabla \vec{S}$ as

$$\nabla \vec{S} = \vec{S}\,\vec{n}\; K_s + \vec{n}\,\vec{n}\,\theta_{ns} + \frac{\vec{n}\,\vec{b}}{2}(\Omega_s + \psi_s) + \frac{\vec{b}\,\vec{n}}{2}(\psi_s - \Omega_s) \qquad (13)$$

From this we can obtain rotational and div of the vector field $\vec{S}$ as

$$\text{curl}\ \vec{S} = \Omega_s\, \vec{S} + K_s\, \vec{b} \qquad (14)$$

$$\text{div}\ \vec{S} = \theta_{ns} + \theta_{bs} = J \qquad (15)$$

where J and Ω_s are the mean curvature and the abnormality of the vector $\vec{S}$.

Following Marris & Passman[5] we have the following geometric results pertaining to the $\vec{n}$ and $\vec{b}$ lines

$$\text{curl}\ \vec{n} = \Omega_n\, \vec{n} + \vec{b}\,\theta_{ns} - \vec{S}\,\theta_{nb} = \Omega_n\, \vec{n} + K_n\, \vec{n_b} \qquad (16)$$

$$\text{div}\ \vec{n} = \theta_{bn} - K_s \qquad (17)$$

$$\text{curl}\ \vec{b} = \vec{S}\,\theta_{bs} + \vec{b}\,\Omega_b - \vec{n}\,\theta_{bs} = \Omega_b\, \vec{b} + K_b\, \vec{b_b} \qquad (18)$$

$$\text{div}\ \vec{b} = \theta_{nb} \qquad (19)$$

where $\Omega_n = -\{\tau_s + \frac{1}{2}(\psi_s - \Omega_s)\}$, $\Omega_b = \frac{1}{2}(\psi_s + \Omega_s) - \tau_s$, $(K_n, \vec{n_b})$ and $(K_b, \vec{b_b})$ are the abnormalities, the curvature and the binormals for the curves of congruences having $\vec{n}$ and $\vec{b}$ as the unit tangent vectors respectively.

Making use of solenoidal property of $\vec{s}$, $\vec{n}$ and $\vec{b}$, we obtain successively the following

$$\frac{\delta\,\Omega_s}{\delta s} - \frac{\delta K_s}{\delta b} + \Omega_s(\theta_{ns} + \theta_{bs}) + K_s\,\theta_{nb} = 0 \qquad (20)$$

$$\frac{\delta\,\Omega_n}{\delta n} + \frac{\delta\,\theta_{ns}}{\delta b} - \frac{\delta\,\theta_{nb}}{\delta s} - \theta_{bs}\theta_{nb} + \Omega_n(\theta_{bn} - K_s) = 0 \qquad (21)$$

$$\frac{\delta\,\theta_{bn}}{\delta s} - \frac{\delta\,\theta_{bs}}{\delta n} + \frac{\delta\,\Omega_b}{\delta b} + \Omega_b\,\theta_{nb} + K_s\,\theta_{bs} + \theta_{bn}\,\theta_{ns} = 0 \qquad (22)$$

Using the irrotational property of ∇f, where $f(\vec{r})$ is the scalar point function in space and equating the corresponding components we obtain the following

$$\frac{\delta^2 f}{\delta b\,\delta n} - \frac{\delta^2 f}{\delta n\,\delta b} = \Omega_s\frac{\delta f}{\delta s} + \theta_{bn}\frac{\delta f}{\delta b} - \theta_{nb}\frac{\delta f}{\delta n} \qquad (23)$$

$$\frac{\delta^2 f}{\delta s\,\delta b} - \frac{\delta^2 f}{\delta b\,\delta s} = \Omega_n\frac{\delta f}{\delta n} - \theta_{bs}\frac{\delta f}{\delta b} \qquad (24)$$

$$\frac{\delta^2 f}{\delta n\, \delta s} - \frac{\delta^2 f}{\delta s\, \delta n} = K_s \frac{\delta f}{\delta s} + \theta_{ns} \frac{\delta f}{\delta n} + \Omega_b \frac{\delta f}{\delta b} \tag{25}$$

From these we observe that the commutative character of intrinsic derivatives do not hold good as in the case of continuous partial derivatives.

INTRINSIC PROPERTIES OF FLOWS

We transform the basic equations into intrinsic form, using the geometric results of the previous section pertaining to the spatial streamlines and the fieldlines and study the kinematic and kinetic properties of the flows described.

Introducing the triad $(\vec{s}, \vec{n}, \vec{b})$ along the lines of force and their principal normals and binormals the momentum equation (2) can be decomposed into intrinsic from as

$$\rho q \frac{\delta q}{\delta s} + \frac{\delta}{\delta s}\left(p + \frac{\omega^2}{2}\right) = \omega_s \frac{\delta \omega}{\delta s'} + K_s' \omega^2 \tag{26}$$

$$\rho K_s q^2 + \frac{\delta}{\delta_n}\left(p + \frac{\omega^2}{2}\right) = \omega_n \frac{\delta \omega}{\delta s'} + K_n' \omega^2 \tag{27}$$

$$\frac{\delta}{\delta b}\left(p + \frac{\omega^2}{2}\right) = \omega_b \frac{\delta \omega}{\delta s'} + K_b' \omega^2 \tag{28}$$

where $\frac{\delta}{\delta s'}$, $(\omega_s, \omega_n, \omega_b)$ and (K'_s, K'_n, K'_b) are the directional derivative along a magnetic line of force, the resolutes of magnetic field and the curvature vectors of the lines of force along the streamlines, the principal normals and binormals respectively. These are the equations of motion in which properties of the streamlines and fieldlines are distinguished from the other properties of the flow. The equation (26) has been derived by Wasserman[1]. Also the equations governing non-magnetic flows can be deduced from the above.

Multiplying (1) by q and introducing the kinetic energy density $V = \rho q^2$ and using (15), we obtain

$$\frac{\delta}{\delta s} \log V + (\theta_{ns} + \theta_{bs}) = \frac{1}{2} \frac{\delta}{\delta s} \log q^2 \tag{29}$$

This expresses the mean curvature of the streamline. If the normal congruences of the streamlines are minimal, (29) yields

$$\rho q = \frac{V}{q} = c_0 \tag{30}$$

where c_0 is constant along a streamline. The physical interpretation of this is the momentum per unit mass of a gas is conserved along an individual streamline if the normal congruences of it form minimal[3].
This result also hold for non-magnetic flows as well, since this is independent of the field.

Introducing the kinetic energy density, the magnetic energy density and the total pressure

$$\text{(a) } V = \rho q^2 \qquad \text{(b) } W = \frac{\omega^2}{2} \qquad \text{(c) } Q = p + W \tag{31}$$

The momentum equation (2) can be written as

$$(\vec{s}.\ \nabla V)\vec{s} + V\{\vec{s}(\theta_{ns} + \theta_{bs}) + K_s \vec{n}\} + \nabla Q = 2W\{K' \vec{n}' - (\theta_{n's'} + \theta_{b's'})\vec{s}'\} \tag{32}$$

Now decomposing these into intrinsic form along a streamline, its principal normal and binormal we obtain

$$\frac{\delta V}{\delta s} + V(\theta_{ns} + \theta_{bs}) + \frac{\delta Q}{\delta s} = 2\{W K'_s - (\theta_{n's'} + \theta_{b's'})\ W_s'\} \tag{33}$$

$$V K_s + \frac{\delta Q}{\delta n} = 2\{W K'_n - (\theta_{n's'} + \theta_{bs''}) W_n\}. \tag{34}$$

$$\frac{\delta Q}{\delta b} = 2\{W K'_b - (\theta_{n's'} + \theta_{b's'}) W_b'\} \tag{35}$$

These are the equations governing the principle of conservation of momentum intrinsic form.

Making use of the above geometric results, the field equation (3) can be decomposed as,

$$q\left(\omega_b\,\theta_{nb}+\omega_n\,\theta_{bn}\right)+\frac{\delta}{\delta n}\left(q\,\omega_n\right)+\frac{\delta}{\delta b}\left(q\,\omega_b\right)=0 \tag{36}$$

$$q\left(\omega_b\,\Omega_n+\omega_n\,\theta_{bs}\right)+\frac{\delta}{\delta s}\left(q\,\omega_n\right)=0 \tag{37}$$

$$q\left(\omega_b\,\theta_{ns}-\omega_n\,\Omega_b\right)+\frac{\delta}{\delta s}\left(q\,\omega_b\right)=0 \tag{38}$$

Irrotational character of the electric field yields

$$E_s\,\Omega_s-E_n\,\theta_{nb}+E_b\,\theta_{bn}+\frac{\delta E_b}{\delta n}-\frac{\delta E_n}{\delta b}=0 \tag{39}$$

$$E_n\,\Omega_n-E_b\,\theta_{bs}+\frac{\delta E_s}{\delta b}-\frac{\delta E_b}{\delta s}=0 \tag{40}$$

$$E_s\,K_s+E_n\,\theta_{ns}+E_b\,\Omega_b+\frac{\delta E_n}{\delta s}-\frac{\delta E_s}{\delta n}=0 \tag{41}$$

where E_s, E_n and E_b are the resolutes of electric field along a streamline, its principal normal and binormal respectively.

Introducing the velocity of sound and mach number in continuity equation (1) can be written as

$$-\left(\theta_{ns}+\theta_{bs}\right)=\frac{\delta}{\delta s}\log M+\frac{\gamma+1}{\gamma-1}\,\frac{\delta}{\delta s}\log c$$

$$=\frac{\delta}{\delta s}\log q+\frac{2}{\gamma-1}\,\frac{\delta}{\delta s}\log c \tag{42}$$

Also the momentum equation (2) can be written as

$$(\vec{q}\,.\,\nabla)\,\vec{q}+\frac{1}{\rho}\,\nabla\,\frac{\omega^2}{2}+\frac{\nabla\,c^2}{\gamma-1}=\frac{c^2}{Jc_p\,(\gamma-1)}\,\nabla S+\frac{1}{\rho}\,(\vec{\omega}\,.\,\nabla)\,\vec{\omega} \tag{43}$$

This can be decomposed into intrinsic form as

$$q\,\frac{\delta q}{\delta s}+\frac{1}{2\rho}\,\frac{\delta\omega^2}{\delta s}+\frac{1}{\gamma-1}\,\frac{\delta c^2}{\delta s}=\frac{1}{\rho}\left(\omega_s\,\frac{\delta\omega}{\delta s'}+K'_s\,\omega^2\right) \tag{44}$$

$$K_s\,q^2+\frac{1}{2\rho}\,\frac{\delta\omega^2}{\delta n}+\frac{1}{\gamma-1}\,\frac{\delta c^2}{\delta n}=\frac{c^2}{Jc_p\,(\gamma-1)}\,\frac{\delta S}{\delta n}+\frac{1}{\rho}\left(\omega^2\,K'_n+\omega_n\,\frac{\delta\omega}{\delta s'}\right) \tag{45}$$

$$\frac{1}{2\rho}\,\frac{\delta\omega^2}{\delta b}+\frac{1}{\gamma-1}\,\frac{\delta c^2}{\delta b}=\frac{c^2}{Jc_p\,(\gamma-1)}\,\frac{\delta S}{\delta b}+\frac{1}{\rho}\left(K'_b\,\omega^2+\omega_b\,\frac{\delta\omega}{\delta s'}\right) \tag{46}$$

These constitute the momentum equations in intrinsic form.

The conservation principle of the magnetic-field can be expressed in fieldline and streamline geometrics respectively as

$$\left(\theta_{n's'}+\theta_{b's'}\right)+\frac{\delta}{\delta s'}\log\omega=0 \tag{47}$$

$$\frac{\delta\,\omega_s}{\delta s}+\frac{\delta\,\omega_n}{\delta n}+\frac{\delta\,\omega_b}{\delta b}+\omega_s\left(\theta_{ns}+\theta_{bs}\right)+\omega_n\left(\theta_{bn}-K_s\right)+\omega_b\,\theta_{nb}=0 \tag{48}$$

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