# THERMOELASTIC CONTACT PROBLEM OF AN ELASTIC LAYER RESTING ON AN ELASTIC FOUNDATION

K. N. SRIVASTAVA & O. P. GUPTA

Maulana Azad College of Technology, Bhopal

(*Received* 17 *March* 1971 ; *revised* 25 *June* 1971)

This paper gives an analysis of the distribution of thermal stress in an infinite isotropic elastic layer, which rests on a semi-infinite isotropic elastic foundation and is indented by a rigid heated punch. The thermal and elastic properties of the layer and foundation are assumed to be different. Two problems are discussed. In the first problem the punch is a flat ended circular cylinder of unit radius, while in the second it is of conical shape. The problems are first reduced to dual integral equations, which are further reduced to two Fredholm integral equations of the second kind. Iterative solutions of these equations are obtained for large value of $h$. Expressions for quantities of physical interest are derived.

George & Sneddon[1] have discussed an axially symmetric problem of elastic half-space indented by a heated punch and made a comparision between the stresses caused by the punch alone and the stresses induced by thermal effects. Using a different technique Keer & Fu[2] considered stress distribution in an elastic plate due to heated punch. Recently Dhaliwal[3] has considered a punch problem for an elastic layer lying over an elastic foundation.

In this paper we shall study distribution of thermal stresses in an elastic layer indented by a rigid heated punch and resting on an elastic foundation. We shall consider two problems. In the first problem the punch is a flat ended circular cylinder while in the second it is of conical shape.

By making a suitable representation of the temperature function, the heat conduction problem is reduced to the solution of a Fredholm integral equation of the second kind. Then using the solution of thermoelastic displacement differential equation, the problem is reduced to the solution of similar Fredholm integral equation in which the solution of the earlier integral equation arising from the heat conduction problem occurs as a known function. Iterative solutions of the integral equations are found, which are valid for large values of $h$. These solutions are used for deriving expressions for quantities of physical interest.

## FORMULATION OF THE PROBLEM

Consider an infinite, isotropic homogeneous elastic layer included between the planes $z = 0$ and $z = -h$ of a cylindrical co-ordinate system $(r, \theta, z)$. The semi-infinite isotropic homogeneous space $z \geqslant 0$ is an elastic foundation upon which the layer rests. The thermal and elastic properties of the layer and of the foundation are assumed to be different. In the case of symmetrical deformation the displacement vector $U$ assumes the form $(u_r, 0, u_z)$ and the non-vanishing components of stress tensor will be $\sigma_{rr}$, $\sigma_{\phi\phi}$, $\sigma_{zz}$, and $\sigma_{rz}$. The region is divided into two domains (1) the layer $-h \leqslant z < 0$ and (2) the semi-infinite elastic space $0 < z \leqslant \infty$.

The boundary conditions at the free surface $z = -h$, when it is indented by the rigid heated punch are :

$$u_z^{(1)}(r, -h) = g(r) \qquad 0 \leqslant r < 1 \tag{1}$$

$$\sigma_{zz}^{(1)}(r, -h) = 0 \qquad r > 1 \tag{2}$$

$$\sigma_{rz}^{(1)}(r, -h) = 0 \qquad r \geqslant 0 \tag{3}$$

the function $g(r)$ can be expressed according to the shape of the punch.

We shall consider the following two cases of temperature conditions at the free surface $z = -h$ :

*Case* (*a*)

When the temperature is prescribed,

$$\left.\begin{aligned} T^{(1)}(r, -h) &= T_1(r) \qquad 0 \leqslant r < 1 \\ &= 0 \qquad\qquad r > 1 \end{aligned}\right\} \tag{4}$$

*Case* (*b*)

When the flux of heat is prescribed,

$$\left.\frac{\partial T^{(1)}}{\partial z}\right|_{z=-h} = T_1(r) \qquad 0 \leqslant r < 1 \tag{5}$$

$$T^{(1)}(r, -h) = 0 \qquad r > 1 \tag{6}$$

Since the elastic layer is perfectly in contact with elastic foundation, the continuity conditions on the interface $z = 0$ are

$$u_r^{(1)}(r, 0) = u_r^{(2)}(r, 0) \tag{7}$$

$$u_z^{(1)}(r, 0) = u_z^{(2)}(r, 0) \tag{8}$$

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_{zz}^{(2)}(r, 0) \text{ for all values of } r \tag{9}$$

$$\sigma_{rz}^{(1)}(r, 0) = \sigma_{rz}^{(2)}(r, 0) \tag{10}$$

$$T^{(1)}(r, 0) = T^{(2)}(r, 0) \tag{11}$$

$$k_1 \left.\frac{\partial T^{(1)}}{\partial z}\right|_{z=0} = k_2 \left.\frac{\partial T^{(2)}}{\partial z}\right|_{z=0} \tag{12}$$

where $k_1$ and $k_2$ are thermal conductivities of the layer and foundation, respectively.

## EQUILIBRIUM EQUATIONS OF THERMOELASTICITY

The thermoelastic displacement components $U_i$ in the absence of body forces, satisfy the following system of equations[4]:

$$\mu\, U_{i,kk} + (\lambda + \mu)\, U_{k,ki} - T_{,i} = 0 \tag{13}$$

where the temperature field is determined by Laplace equation

$$T_{,kk} = 0 \tag{14}$$

in the steady state, and in the absence of heat sources.

In the particular case of symmetry of temperature and stress fields with respect to $z$-axis, equations (13), (14) reduce to two equations

$$\left.\begin{aligned} \nabla^2 u_r - r^{-2}\, u_r + \frac{1}{1-2\eta}\, e_{,r} - \frac{2(1+\eta)}{1-2\eta}\,\alpha_t\, T_{,r} &= 0 \\ \nabla^2 u_z + \frac{1}{1-2\eta}\, e_{,z} - \frac{2(1+\eta)}{1-2\eta}\,\alpha_t\, T_{,z} &= 0 \\ \nabla^2 T &= 0 \end{aligned}\right\} \tag{15}$$

where

$$e = u_{r,r} + r u_r + u_{z,z} \qquad \nabla^2 = \partial^2 r + r^{-1}\partial r + \partial z$$

$\mu$ is modulus of rigidity, $\eta$ Poisson's ratio, $\alpha_t$ the coefficient of linear expansion.

The two components of stress, in terms of displacements are

$$\sigma_{rz}(r, z) = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{16}$$

$$\sigma_{zz}(r, z) = \frac{2\mu}{1-2\eta}\left[(1-\eta)\frac{u_z}{\partial z} + \eta\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) - (1+\eta)\,\alpha_t\, T\right] \tag{17}$$

In region 1 ($-h \leqslant z < 0$) we follow the method given by Srivastava & Palaiya[5] for deriving the expressions for the displacements, stresses and the temperature field. These expressions are

$$2\mu_1 u_r^{(1)} = \int_0^\infty \xi \left[ (C + DZ) e^{\xi z} + (E + FZ) e^{-\xi z} \right] J_0' (\xi r) \, d\xi \tag{18}$$

$$2\mu_1 u_z^{(1)} = \int_0^\infty \left[ \left\{ \xi (C + DZ) - (3 - 4\eta_1) D + 2A \right\} e^{\xi z} - \right.$$

$$\left. - \left\{ \xi (E + FZ) - (3 - 4\eta_1) F + 2B \right\} e^{-\xi z} \right] J_0 (\xi r) \, d\xi \tag{19}$$

$$\sigma_{zz}^{(1)} = \int_0^\infty \left[ \left\{ \xi (C + DZ) - (2 - 2\eta_1) \; D + A \right\} e^{\xi z} + \right.$$

$$\left. + \left\{ \xi (E + FZ) + (2 - 2\eta_1) \; F + B \right\} e^{-\xi z} \right] \xi J_0 (\xi r) \, d\xi \tag{20}$$

$$\sigma_{rz}^{(1)} = \int_0^\infty \left[ \left\{ \xi (C + DZ) - (1 - 2\eta_1) D + A \right\} e^{\xi z} - \right.$$

$$\left. - \left\{ \xi (E + FZ) + (1 - 2\eta_1) F + B \right\} e^{-\xi z} \right] \xi J_0' (\xi r) \, d\xi \tag{21}$$

$$T^{(1)} = \frac{1}{\alpha_1 (1 + \eta_1)} \int_0^\infty \left[ A e^{\xi z} + B e^{-\xi z} \right] \xi J_0 (\xi r) \, d\xi \tag{22}$$

where $A, B, C, \ldots \ldots E$ and $F$ are unknown constants to be determined.

The corresponding expressions for the region 2 ($z > 0$), which are obtained by replacing $\alpha_1$, $\mu_1$, $\eta_1$, $E$, $F$ and $B$ by $\alpha_2$, $\mu_2$, $\eta_2$, $E_1$, $F_1$ and $B_1$ respectively and putting $A = C = D = 0$ in the above expressions, are :

$$2\mu_2 u_r^{(2)} = \int_0^\infty \xi (E_1 + F_1 Z) e^{-\xi z} J_0' (\xi r) \, d\xi \tag{23}$$

$$2\mu_2 u_z^{(2)} = - \int_0^\infty \left[ \xi (E_1 + F_1 Z) + (3 - 4\eta_2) F_1 + 2B_1 \right] e^{-\xi z} J_0 (\xi r) \, d\xi \tag{24}$$

$$\sigma_{zz}^{(2)} = \int_0^\infty \left[ \xi (E_1 + F_1 Z) + (2 - 2\eta_2) F_1 + B_1 \right] e^{-\xi z} \xi J_0 (\xi r) \, d\xi \tag{25}$$

$$\sigma_{rz}^{(2)} = - \int_0^\infty \left[ \xi (E_1 + F_1 Z) + (1 - 2\eta_2) F_1 + B_1 \right] e^{-\xi z} \xi J_0' (\xi r) \, d\xi \tag{26}$$

$$T^{(2)} = \frac{1}{\alpha_2 (1 + \eta_2)} \int_0^\infty \xi B_1 e^{-\xi z} J_0 (\xi r) \, d\xi \tag{27}$$

where

$$\mu_i = E_i'/2(1+\eta_i), \qquad i = 1, 2$$

and $E_i'$ is the Young's modulus.

## TEMPERATURE FIELDS

Let us consider both the cases of temperature fields.

The conditions (11) and (12) are satisfied, provided

$$A = \tfrac{1}{2}(1-k)\,\frac{\alpha_1(1+\eta_1)}{\alpha_2(1+\eta_2)}\,B_1$$

$$B = \tfrac{1}{2}(1+k)\,\frac{\alpha_1(1+\eta_1)}{\alpha_2(1+\eta_2)}\,B_1$$

Hence (22) can be written as

$$T^{(1)} = \frac{1}{2\,\alpha_2(1+\eta_2)\,k_1}\int_0^\infty \xi\,B_1(\xi)\left[(k_1-k_2)\,e^{\xi z} + (k_1+k_2)\,e^{-\xi z}\right] J_0(\xi r)\,d\xi \tag{28}$$

where

$$k = k_1/k_2$$

*Case (a)*

Thus case (a) gives the following equations

$$\left.\begin{aligned}\int_0^\infty \xi\,B_1(\xi)\left[(k_1-k_2)\,e^{-\xi h} + (k_1+k_2)\,e^{\xi h}\right] J_0(\xi r)\,d\xi &= T_1'(r) \qquad 0 \leqslant r < 1\\ &= 0 \qquad r > 1\end{aligned}\right\} \tag{29}$$

or we have

$$\left[(k_1-k_2)\,e^{-\xi h} + (k_1+k_2)\,e^{\xi h}\right] B_1(\xi) = \int_0^1 r\,T_1'(r)\,J_0(\xi r)\,dr \tag{30}$$

where

$$T_1'(r) = 2\alpha_2(1+\eta_2)\,k_1\,T_1(r).$$

*Cas (b)*

Similarly for the case (b) we obtain the following pair of dual integral equations

$$\left.\begin{aligned}\int_0^\infty \left[1 + H'(2\xi h)\right] \xi\,\phi(\xi)\,J_0(\xi r)\,d\xi &= T_1''(r) \qquad 0 \leqslant r < 1\\ \int_0^\infty \phi(\xi)\,J_0(\xi r)\,d\xi &= 0 \qquad r > 1\end{aligned}\right\} \tag{31}$$

where

$$H'(2\xi h) = \frac{2a'\,e^{-2\xi h}}{1 - a'\,e^{-2\xi h}} \qquad \text{and} \qquad a' = \frac{k_2 - k_1}{k_2 + k_1}$$

and

$$T_1''(r) = -2\alpha_2(1+\eta_2)k_1 T_1(r)$$

$$\phi(\xi) = \xi B_1(\xi)\left[(k_1-k_2)e^{-\xi h}+(k_1+k_2)e^{\xi h}\right] \tag{32}$$

The solution of the above dual integral equations as given by Sneddon[6] is

$$\phi(\xi) = \int_0^1 g'(t)\sin(\xi t)\,dt \tag{33}$$

where $g'(0) = 0$

and $g'(t)$ is determined from the Fredholm integral equation

$$g'(t) = A(t) - \int_0^1 g'(t)K'(S,t)\,dS \tag{34}$$

where $$A(t) = \frac{2}{\pi}\int_0^t \frac{r\,T_1''(r)\,dr}{(t^2-r^2)^{\frac{1}{2}}} \tag{35}$$

$$K'(S,t) = \frac{2}{\pi}\int_0^\infty H'(2\xi h)\sin(\xi t)\sin(\xi S)\,d\xi$$

$$= \frac{1}{\pi}\left[\frac{2tS}{h^3}I_1 - \frac{tS(t^2+S^2)}{3h^5}I_2 + O(h^{-7})\right] \tag{36}$$

and

$$I_n = \int_0^\infty \lambda^{2n}H'(2\lambda)\,d\lambda, \quad n=1,2 \tag{37}$$

We assume that the iterative solution of (34) is

$$g'(t) = g_0'(t) + \frac{g_1'(t)}{h} + \ldots + \frac{g_5'(t)}{h^5} + O(h^{-6}) \tag{38}$$

If we take $T_1''(r)$ = constant = $\theta_0$ (say) then,

$$g_0'(t) = \frac{2}{\pi}\theta_0 t = \theta t\,(\text{say})$$

$$g_1'(t) = g_2'(t) = 0$$

$$g_3'(t) = -\frac{2I_1 t}{\pi}\int_0^1 S\,g_0'(S)\,dS = -\frac{2I_1}{3\pi}\theta t$$

$$g_4'(t) = -\frac{2I_1 t}{\pi}\int_0^1 S\, g_1'(S)\, dS = 0$$

$$g_5'(t) = -\frac{1}{\pi}\int_0^1 \left[ 2I_1\, t\, S\, g_2'(S) - \frac{t\, S\,(t^2+S^2)}{3}\, I_2\, g_0'(S) \right] dS$$

$$= \frac{I_2\, \theta}{45\pi}\, t\,(3+5t^2)$$

so that

$$g'(t) = \theta\, t \left[ 1 - \frac{2I_1}{3\pi\, h^3} + \frac{I_2}{45\pi\, h^5}\,(3+5t^2) + O\,(h^{-7}) \right] \tag{39}$$

where

$$\theta = 2/\pi\, \theta_0 \,.$$

## REDUCTION OF PROBLEM TO DUAL INTEGRAL EQUATIONS

The continuity conditions (7) to (10) are satisfied, provided

$$4\,(1-\eta_1)\,\xi\, C = (\mu-1)\,(3-4\eta_1)\,\xi\, E_1 + \alpha F_1 + \beta B_1 \tag{40}$$

$$4\,(1-\eta_1)\, D = 2\,(\mu-1)\,\xi\, E_1 + (\mu-1)\,(3-4\eta_2)\, F_1 + \nu\, B_1 \tag{41}$$

$$4\,(1-\eta_1)\,\xi\, E = (\mu+3-4\eta_1)\,\xi\, E_1 - \alpha F_1 - \beta B_1 \tag{42}$$

$$4\,(1-\eta_1)\, F = [1+\mu\,(3-4\eta_2)]\, F_1 + \alpha_0\, B_1 \tag{43}$$

where

$$\alpha = \mu\,(3-4\eta_2)\,(1-2\eta_1) - (1-2\eta_2)\,(3-4\eta_1)$$

$$\alpha_0 = 2\mu - (1+k)\, L$$

$$\beta = \mu\,(2-4\eta_1) - 3 + 4\eta_1 + Lk$$

$$\nu = 2\mu - 2 + (1-k)\, L$$

$$L = \alpha_1\,(1+\eta_1)\,/\,\alpha_2\,(1+\eta_2)$$

and $\mu = \mu_1/\mu_2$

Now the application of (3) leads to

$$\xi\, E_1 \left[ (\mu-1)\,(1-2\xi\, h)\, e^{-2\xi h} - \mu - 3 + 4\eta_1 \right]$$

$$= -F_1 \left[ \left\{ \alpha - (\mu-1)\,(3-4\eta_2)\,(\xi\, h + 1 - 2\eta_1) \right\} e^{-2\xi h} - \right.$$

$$\left. - (1-2\eta_1-\xi h) \left\{ 1 + \mu\,(3-4\eta_2) \right\} + \alpha \right] -$$

$$- B_1 \left[ (L-1-\nu\xi h)\, e^{-2\xi h} - L - 3 + 4\eta_1 + \alpha_0\, \xi h \right] \tag{44}$$

Equations (40) to (43) express the unknown functions $C(\xi)$, $D(\xi)$, $E(\xi)$ and $F(\xi)$ in terms of $E_1(\xi)$, $F_1(\xi)$ and $B_1(\xi)$ and (44) expresses the unknown function $E_1(\xi)$ in terms of single unknown function $F_1(\xi)$.

Now, if we define $F_1(\xi)$ in terms of another unknown function $G(\xi)$ by the relation

$$\frac{F_1(\xi)}{4(1-\eta_1)} = -\frac{\mu'(1-x)e^{-x} - M_3}{M_2 M_3 \chi(x)} e^{-\xi h} G(\xi) - \frac{B_1(\xi)[\mu' \alpha_0 e^{-2x} + \alpha_{11} e^{-x} + M_3 \alpha_0]}{M_2 M_3 \chi(x)} \tag{45}$$

where

$$x = 2\xi h, \qquad M_1 = (3-4\eta_1) - \mu(3-4\eta_2)$$

$$M_2 = 1 + \mu(3-4\eta_2), \qquad M_3 = \mu + 3 - 4\eta_1$$

$$\mu' = \mu - 1 \qquad \alpha_{11} = 8L(1-k)(1-\eta_1)$$

$$\chi(x) = 1 + [a + b(1-x^2)]e^{-x} + c e^{-2x}$$

and

$$a = M_1/M_2, \qquad b = -\mu'/M_3, \qquad c = -\mu' M_1/M_2 M_3,$$

the boundary conditions (1) and (2) are satisfied if $G(\xi)$ is the solution of the dual integral equations

$$\int_0^\infty \left[ 1 + H(2\xi h) \right] G(\xi) J_0(\xi r) d\xi = -\frac{G'(r)}{2-2\eta_1} \qquad 0 \leqslant r < 1 \tag{46}$$

$$\int_0^\infty \xi G(\xi) J_0(\xi r) d\xi = 0 \qquad r > 1 \tag{47}$$

where

$$H(x) = -\frac{e^{-x}[a + b(1+x^2) + 2c e^{-x}]}{\chi(x)} \tag{48}$$

and $G'(r)$ is defined as

$$G'(r) = g(r) + \frac{k_1}{M_3(k_2+k_1)} \int_0^\infty \frac{\phi(\xi)}{\xi} \left[ 1 + Q(2\xi h) \right] J_0(\xi r) d\xi \tag{49}$$

$\phi(\xi)$ can be obtained from equation (33) and $Q(x)$ is given as

$$Q(x) = \Big[ e^{-4x}(\theta_1 - c P_2 x) + e^{-3x}(\theta_2 + \theta_3 x + b P_1 x^2 - b P_2 x^3) + e^{-2x}(\theta_4 + \theta_5 x - b P_3 x^2 - b P_4 x^3) + e^{-x}(\theta_6 + \theta_7 x + b P_5 x^2) \Big] \div \Big[ \Big\{ 1 + b(1-x)e^{-x} \Big\} . (1 - a' e^{-x}) \chi(x) \Big] \tag{50}$$

where

$$P_1 = \frac{\mu'(L-\mu)}{K} + a' b, \qquad P_2 = \frac{\mu' L(1-k)}{2K} + a' b$$

$$P_3 = \frac{\mu' L}{K} - a' + b, \qquad P_4 = \frac{\mu'[L(1+k) - 4\mu]}{2K} - b$$

$$P_5 = \frac{\mu M_3}{K} - 1$$

and

$$\theta_1 = P_1 + \frac{\mu' \alpha_0 (2 - 2\eta_1)}{K}$$

$$\theta_2 = P_1 (a + b) - c \left[ \frac{\mu' L}{K} + \frac{16L (1 - k)(1 - \eta_1)^2}{K} - a' + b \right]$$

$$\theta_3 = 2b \frac{\mu' \alpha_0 (2 - 2\eta_1)}{K} - c P_4 - (a + b) P_2$$

$$\theta_4 = P_1 + c P_5 - (\mu' - c M_3) \frac{\mu' \alpha_0 (2 - 2\eta_1)}{K} - (a + b) P_3$$

$$\theta_5 = 32b \frac{L (1 - k)(1 - \eta_1)^2}{K} - P_2 - (a + b) P_4$$

$$\theta_6 = (a + b) P_5 - \left[ \frac{\mu' L}{K} + \frac{16L (1 - k)(1 - \eta_1)^2}{K} - a' + b \right]$$

$$\theta_7 = 2b M_3 \frac{\mu' \alpha_0 (2 - 2\eta_1)}{K} - P_4$$

where $K = \alpha_0 \left[ 1 - (2 - 2\eta_1) M_3 \right]$.

The solution of the dual integral equations (46) and (47) as given by Sneddon[6] is

$$G(\xi) = \int_0^1 W(t) \cos(\xi t)\, dt \tag{51}$$

$W(t)$ is determined from the Fredholm integral equation

$$W(t) = p_1 \psi(t) + p_1 p_2 \int_0^\infty \frac{\phi(\xi)}{\xi} \left[ 1 + Q(2\xi h) \right] \cos \xi t\, dt - \int_0^1 K(S, t)\, W(S)\, dS \tag{52}$$

where $p_1 = -\dfrac{2}{\pi (2 - 2\eta_1)}$ $\qquad p_2 = \dfrac{k_1}{M_3 (k_2 + k_1)}$

$$\psi(t) = \frac{d}{dt} \int_0^t \frac{r\, g(r)\, dr}{(t^2 - r^2)^{\frac{1}{2}}} \tag{53}$$

$$K(S, t) = \frac{2}{h \pi} \int_0^\infty H(2\lambda) \cos(\lambda S/h) \cos(\lambda t/h)\, d\lambda$$

$$= \frac{1}{\pi} \left[ \frac{j_0}{h} - \frac{S^2 + t^2}{h^3} j_1 + \frac{t^4 + 6t^2 S^2 + S^4}{h^5} j_2 + O(h^{-7}) \right] \tag{54}$$

and
$$j_n = \frac{1}{2n\,!} \int_0^\infty H(2\lambda)\, \lambda^{2n}\, d\lambda \quad (n = 0, 1, 2) \tag{55}$$

The expression for $K(S, t)$ is obtained by substituting $\xi\, h = \lambda$ and writing the expansion for cosine and sine functions in powers of $h$.

On substituting the value of $\phi(\xi)$ from (33) into (52), we get

$$W(t) = p_1 \psi(t) + \frac{p_1 p_2}{2} \pi \int_t^1 g'(u)\, du + p_1 p_2 \int_0^1 g'(u)\, du \int_0^\infty \frac{1}{\xi} Q(2\xi h) \sin(\xi u) \cdot$$
$$\cdot \cos(\xi t)\, d\xi - \int_0^1 K(S, t)\, W(S)\, dS \tag{56}$$

Inserting the value of $g'(u)$ from (39) into (56), we get

$$W(t) = p_1 \psi(t) + \frac{2\, p_1 p_2}{\pi} \theta_0 \left[ \frac{\pi}{4}(1 - t^2) + \frac{\delta_0}{3h} - \frac{1}{30h^3} \left\{ 5I_1 (1 - t^2) + \right.\right.$$
$$\left. + 6\delta_1 (1 + 5t^2) \right\} - \frac{2\delta_0\, I_1}{9\pi\, h^4} + \frac{1}{2520h^5} \left\{ 7I_2\, (11 - 6t^2 - 5t^4) + \right.$$
$$\left.\left. + 120\delta_2 (3 + 42t^2 + 35t^4) \right\} + O(h^{-6}) \right] - \int_0^1 K(S, t)\, W(S)\, dS \tag{57}$$

where

$$\delta_n = \frac{1}{(2n+1)\,!} \int_0^\infty Q(2\lambda)\, \lambda^{2n}\, d\lambda \quad (n = 0, 1, 2) \tag{58}$$

An iterative solution of (57) can be obtained by writing

$$W(t) = W_0(t) + \frac{W_1(t)}{h} + \ldots\ldots + \frac{W_5(t)}{h^5} + O(h^{-6}) \tag{59}$$

where

$$W_0(t) = p_1 \psi(t) + \frac{p_1 p_2}{2} \theta_0 (1 - t^2)$$

$$W_1(t) = \frac{2p_1 p_2}{3\pi} \theta_0\, \delta_0 - \frac{j_0}{\pi} \int_0^1 W_0(S)\, dS \tag{60}$$

$$W_2(t) = - \frac{j_0}{\pi} \int_0^1 W_1(S)\, dS$$

$$W_3(t) = - \frac{p_1 p_2}{15\,\pi} \theta_0 \left[ 5I_1 (1 - t^2) + 6\delta_1 (1 + 5t^2) \right] - \frac{1}{\pi} \int_0^1 \left[ W_2(S)\, j_0 - \right.$$
$$\left. \cdot - j\, W_0(S)\, (S^2 + t^2) \right] dS$$

$$W_4(t) = -\frac{4p_1 p_2}{9\pi^2}\,\delta_0 I_1 \theta_0 - \frac{1}{\pi}\int_0^1 \Big[ W_3(S)\, j_0 - j_1 W_1(S)(S^2+t^2) \Big] dS$$

$$W_5(t) = \frac{p_1 p_2}{1260\pi}\,\theta_0 \Big[ 7I_2 (11 - (t^2 - 5t^4) + 120\delta_2 (3 + 42t^2 + 35t^4) \Big] -$$

$$- \frac{1}{\pi}\int_0^1 \Big[ W_4(S)\, j_0 - W_2(S)\, j_1 (S^2+t^2) + W_0(S)\, j_2 (t^4 + (t^2 S^2 + S^4) \Big] dS$$

Numerical values of $I_n$ $(n = 1, 2)$, $\delta_n$ $(n = 0, 1, 2)$ have been computed for $\eta_1 = .33$, $\eta_2 = .25$, $k_1 = .42$ $k_2 = .54$, $E_1' = 10\times 10^{11}$ dyn/cm², $E_2' = 21\times 10^{11}$ dyn/cm², $\alpha_1 = 0\cdot000012$ and $\alpha_2 = 0\cdot0000102$ and $\mu =$

The values of the integral $j_n$ $(n = 0, 1, 2)$ have been[3] given in Table 1.

## NORMAL STRESS UNDER THE PUNCH

We shall now derive the expression for the normal stress under the punch.

The normal stress under the punch is

$$\overset{(1)}{\sigma_{zz}} (r, -h) = \int_0^\infty \xi\, G\ (\xi)\, J_0(\xi r)\, d\xi \qquad (61)$$

Substituting the value of $G(\xi)$ from (51) into (61) and interchanging the order of integration, we get

$$\overset{(1)}{\sigma_{zz}} (r, -h) = -\frac{1}{r}\,\frac{d}{dr}\int_r^1 \frac{t\,W(t)\,dt}{(t^2 - r^2)^{\frac{1}{2}}} \qquad (62)$$

## TOTAL LOAD ON THE PUNCH

We derive the expression of total load $P$ applied by the punch to maintain the displacement.

The total load under the punch is

$$P = -2\pi \int_0^1 \overset{(1)}{\sigma_{zz}} (r, -h)\, r\, dr \qquad (63)$$

inserting the value of $\overset{(1)}{\sigma_{zz}} (r, -h)$ from (62) into (63) we get

$$P = -2\pi \int_0^1 W(t)\, dt \qquad (64)$$

TABLE 1

VALUES OF INTEGRALS $I_n$, $\delta_n$ AND $j_n$ FOR $n=0$, 1 AND 2

| Integrals | $n$ | | |
|---|---|---|---|
| | 0 | 1 | 2 |
| $I_n$ | .. | 0·029632 | 0·088248 |
| $\delta_n$ | 1·191022 | 0·185111 | 0·057212 |
| $j_n$ | 1·246991 | 0·800334 | 0·381609 |

## SHAPE OF THE DEFORMED SURFACE

The shape of the deformed surface for $r > 1$ is

$$\overset{(1)}{u_z}(r, -h) = -(2 - 2\eta_1) \int_0^\infty [1 + H(2\xi h)] \, G(\xi) \, J_0(\xi r) \, d\xi \tag{65}$$

Substituting the value of $G(\xi)$ from (51) into (65) we have

$$\overset{(1)}{u_z}(r, -h) = -(2 - 2\eta_1) \left[ \int_0^1 \frac{W(t)\,dt}{(r^2 - t^2)^{\frac{1}{2}}} + \int_0^1 W(t)\, K(r, t)\, dt \right] \tag{66}$$

where

$$K(r, t) = \frac{1}{h} \int_0^\infty H(2\lambda)\, J_0(\lambda r/h) \cos(\lambda t/h)\, d\lambda$$

$$= \left[ \frac{j_0}{h} - \frac{j_1}{h^3}(t^2 + \tfrac{1}{2}r^2) + \frac{j_2}{h^5}\left( t^4 + \frac{3}{8} r^4 + 3r^2 t^2 \right) \right] + O(h^{-7}) \tag{67}$$

where $j_n$ $(n = 0, 1, 2)$ is defined by the equation (55).

The expression for $K(r, t)$ is obtained by substituting $\xi h = \lambda$ and writing the expansions for cosine and $J_0$ function in powers of $h$.

## PARTICULAR CASES

Here we consider cases for two different shapes of the punch.

*Flat-Ended Cylindrical Punch*

The shape of the punch is a flat ended circular cylinder of unit radius and $\epsilon$ is the depth to which the punch penetrates (see Fig. 1). Then we have

$$g(r) = \epsilon \qquad 0 \leqslant r \leqslant 1 \tag{68}$$

Therefore, the value (52) can be written as

$$\psi(t) = \epsilon \tag{69}$$

Inserting the value of $\psi(t)$ from (69) into the expressions of (60), (59) can be written as

$$W(t) = A_0 + A_1 t^2 + A_2 t^4 + O(h^{-6}) \tag{70}$$

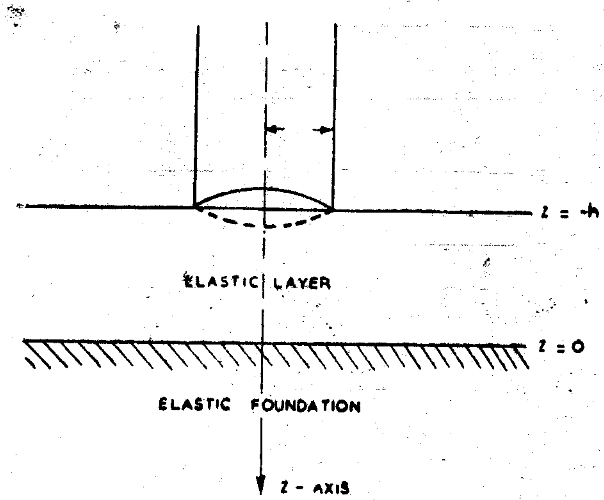

Fig. 1—Flat ended cylindrical punch (problem 1).

where

$$A_0 = \epsilon \left[ -0{\cdot}474898 + \frac{0{\cdot}188501}{h} - \frac{0{\cdot}074822}{h^2} + \frac{0{\cdot}024542}{h^3} + \frac{0{\cdot}036233}{h^4} - \frac{0{\cdot}002845}{h^5} \right] + \theta_0 \left[ -0{\cdot}030365 - \frac{0{\cdot}007314}{h} + \frac{0{\cdot}002902}{h^2} - \frac{0{\cdot}001138}{h^3} - \frac{0{\cdot}000542}{h^4} + \frac{0{\cdot}000506}{h^5} \right]$$

$$A_1 = -\epsilon\left[\frac{0\cdot015471}{h^3}-\frac{0\cdot048021}{h^4}-\frac{0\cdot096309}{h^5}\right]+\theta_0\left[0\cdot030365-\frac{0\cdot001071}{h^3}\right.$$
$$\left.-\frac{0\cdot001863}{h^4}-\frac{0\cdot000634}{h^5}\right]$$

$$A_2 = \epsilon\,\frac{0\cdot057685}{h^5}-\theta_0\,\frac{0\cdot001145}{h^5}$$

The normal stress is

$$\overset{(1)}{\sigma_{zz}}(r,-h) = -(1-r^2)^{-\frac{1}{2}}\left[(A_2-A_1-A_0)-2\left(A_1-\frac{2}{3}A_2\right)r^2-\frac{8}{3}A_2r^4\right]+O(h^{-6}) \quad (71)$$

and the pressure is

$$P = -2\pi\left[A_0+\frac{A_1}{3}+\frac{A_2}{5}\right]+O(h^{-6}) \quad (72)$$

The shape of the deformed surface, i.e. $\overset{(1)}{u_z}(r,-h)$ for $r>1$ is

$$\overset{(1)}{u_z}(r,-h) = -(2-2\eta_1)\left[\left\{S_0+A_0\sin^{-1}1/r-\frac{1}{8}(r^2-1)^{\frac{1}{2}}(4A_1+2A_2)\right\}+\right.$$
$$\left.+\left\{\frac{A_1}{2}\sin^{-1}1/r-\frac{3}{8}A_2(r^2-1)^{\frac{1}{2}}+S_1\right\}r^2+\left\{S_2+\frac{3}{8}A_2\sin^{-1}1/r\right\}r^4\right] \quad (73)$$

where

$$S_0 = A_0L_1+A_1L_4+A_2L_7$$
$$S_1 = A_0L_2+A_1L_5+A_2L_8$$
$$S_2 = A_0L_3+A_1L_6+A_2L_9$$
$$L_1 = \frac{1\cdot246991}{h}-\frac{0\cdot381609}{3h^3}+\frac{0\cdot381609}{5h^5}$$

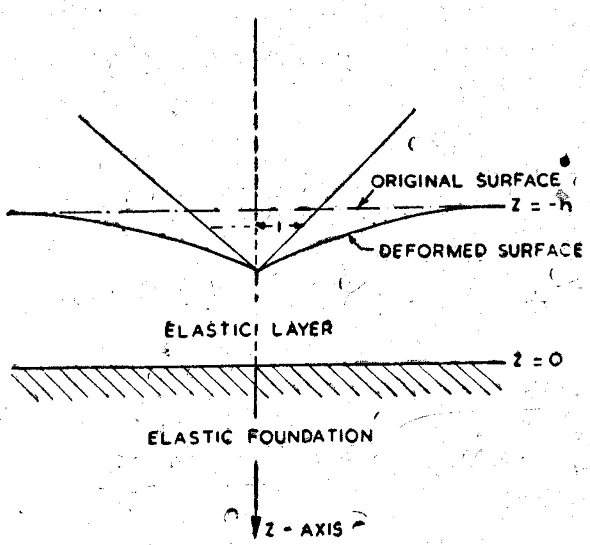


Fig. 2—Conical punch (problem 2).

$$L_2 = -\frac{0\cdot800334}{2h^3}+\frac{0\cdot381609}{h^5};$$
$$L_3 = \frac{3}{8}\cdot\frac{0\cdot381609}{h^3}$$
$$L_4 = \frac{1\cdot246991}{3h}-\frac{0\cdot800334}{5h^3}+\frac{0\cdot381609}{7h^5}$$
$$L_5 = -\frac{0\cdot800334}{6h^3}+\frac{3}{5}\cdot\frac{0\cdot381609}{h^5};$$
$$L_6 = \frac{0\cdot381609}{8h^5}$$
$$L_7 = \frac{1\cdot246991}{5h}-\frac{0\cdot800334}{7h^3}+\frac{0\cdot381609}{9h^5}$$
$$L_8 = -\frac{0\cdot800334}{10h^3}+\frac{3}{7}\cdot\frac{0\cdot381609}{h^5};$$
$$L_9 = \frac{3}{40}\cdot\frac{0\cdot381609}{h^5}$$

*Conical Punch*

The shape of the punch is a right circular cone, and the elastic layer is indented normally by it (see Fig. 2). The displacement is

$$g(r) = \epsilon(\pi/2 - r) \qquad 0 \leq r < 1 \tag{74}$$

Therefore equation (53) is

$$\psi(t) = (\epsilon/2)\,\pi\,(1 - t) \tag{75}$$

Substituting the value of $\psi(t)$ from (75) into (60), equation (59) can be written as

$$W(t) = A_3 + A_4 t + A_5 t^2 + A_6 t^4 + O(h^{-6}) \tag{76}$$

where

$$A_3 = \epsilon\left[-0{\cdot}745968 + \frac{0{\cdot}148049}{h} - \frac{0{\cdot}058765}{h^2} + \frac{0{\cdot}007489}{h^3} + \frac{0{\cdot}022171}{h^4} - \frac{0{\cdot}015760}{h^5}\right] + \theta_0\left[-0{\cdot}030365 - \frac{0{\cdot}007314}{h} + \frac{0{\cdot}002902}{h^2} - \frac{0{\cdot}001138}{h^3} - \frac{0{\cdot}000542}{h^4} + \frac{0{\cdot}000506}{h^5}\right]$$

$$A_4 = 0{\cdot}745968\,\epsilon$$

$$A_5 = -\epsilon\left[\frac{0{\cdot}095022}{h^3} - \frac{0{\cdot}037716}{h^4} - \frac{0{\cdot}030330}{h^5}\right] + \theta_0\left[0{\cdot}030365 - \frac{0{\cdot}001071}{h^3} - \frac{0{\cdot}001863}{h^4} - \frac{0{\cdot}000631}{h^5}\right]$$

$$A_6 = \epsilon\,\frac{0{\cdot}045300}{h^5} - \theta_0\,\frac{0{\cdot}001145}{h^5}$$

The normal stress is

$$\overset{(1)}{\sigma_{zz}}(r, -h) = -(1 - r^2)^{-\frac{1}{2}}\left[(A_6 + A_5 - A_4 - A_3) + \left(\frac{A_4}{2} - 2A_5 + \frac{4}{3}A_6\right) \times r^2 - \frac{8}{3}A_6 r^4 + (1 - r^2)^{-\frac{1}{2}}\cosh^{-1} 1/r\, A_4\right] + O(h^{-6}) \tag{77}$$

and the pressure is

$$P = -2\pi\,[A_3 + A_4/2 + A_5/3 + A_6/5] + O(h^{-6}) \tag{78}$$

The shape of the deformed surface, i.e. $\overset{(1)}{u_z}(r, -h)$ for $r > 1$ is

$$\overset{(1)}{u_z}(r, -h) = -(2 - 2\eta_1)\left[\left\{S_0' + A_3 \sin^{-1} 1/r - 1/8\,(r^2 - 1)^{\frac{1}{2}}\,(8A_4 + 4A_5 + 2A_6)\right\} + A_4 r + \left\{S_1' + \frac{A_5}{2}\sin^{-1} 1/r - 3/8.\,A_2 (r^2 - 1)^{\frac{1}{2}}\right\} r^2 + \{S_2' + 3/8.\,A_6 \sin^{-1} 1/r\}\, r^4\right] \tag{79}$$

where,

$$S_0' = A_3 L_1 + A_4 L_{10} + A_5 L_4 + A_6 L_7$$
$$S_1' = A_3 L_2 + A_4 L_{11} + A_5 L_5 + A_6 L_8$$
$$S_2' = A_3 L_3 + A_4 L_{12} + A_5 L_6 + A_8 L_9$$

and

$$L_{10} = \frac{1{\cdot}246991}{2h} - \frac{0{\cdot}800334}{4h^3} + \frac{0{\cdot}381609}{6h^5}$$

$$L_{11} = -\frac{0 \cdot 800334}{4h^3} + \frac{3}{4} \cdot \frac{0 \cdot 381609}{h^5}$$

$$L_{12} = \frac{3}{16} \cdot \frac{0 \cdot 381609}{h^5}$$

ACKNOWLEDGEMENTS

The work of the first author has been supported by a C.S.I.R. grant. The second author is thankful to the Ministry of Education for a Research Training Scholarship.

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