APPLICATION OF E-OPERATOR IN EVALUATING CERTAIN FINITE INTEGRALS

F. SINGH

Government Engineering College, Rewa

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In an attempt to unify some earlier results in the theory of the functions of hypergeometric type, the author establishes some new results involving the product of generalized hypergeometric function and H-function.

The object of this paper is to evaluate some finite integrals involving the product of generalized hypergeometric function and H-function with the help of finite difference operator E. As on specializing the parameters the generalized hypergeometric function may be converted into a number of special functions and polynomials, and H-function into G-function and all those functions which can be defined through G-function, the integrals obtained in this paper are of general nature and many interesting particular cases have been obtained some of which are known and others are believed to be new.

Bajpai[1], Chhabra & Singh[2], Sharma[3] and Sharma[4] have established useful results in the theory of functions of hypergeometric type. The H-function introduced by Fox[5] is represented and defined as follows :

$$H_{p,\,q}^{K,\,l}\left[x\left|\begin{matrix}(a_1, e_1),\ldots,(a_p, e_p)\\(b_1, f_1),\ldots,(b_q, f_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{K}\Gamma(b_j-f_j S)\prod_{j=1}^{l}\Gamma(1-a_j+e_j S)}{\prod_{j=K+1}^{q}\Gamma(1-b_j+f_j S)\prod_{j=l+1}^{p}\Gamma(a_j-e_j S)}x^S dS, \quad (1)$$

where x is not equal to zero and an empty product is interpreted as unity; p, q, K and l are integers satisfying $1 \leqslant K \leqslant q$, $0 \leqslant l \leqslant p$, $e_j\ (j=1,\ldots,p)$, $f_h\ (h=1,\ldots,q)$ are positive numbers and $a_j\ (j=1,\ldots,p)$, $b_h\ (h=1,\ldots\ldots,q)$ are complex numbers. L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j - f_j S)\ (j=1,\ldots,K)$ ie to the right and those of $\Gamma(1-a_j+e_j S)\ (j=1,\ldots\ldots,l)$ to the left of L. These assumptions for the H-function will be adhered to throughout this paper.

In a more compact form, H-function is represented by

$$H_{p,\,q}^{K,\,l}\left[x\left|\begin{matrix}\{(a_p, e_p)\}\\\{(b_q, f_q)\}\end{matrix}\right.\right]$$

where (α_p, β_p) stands for a set of parameters $(\alpha_1, \beta_1), \ldots\ldots, (\alpha_p, \beta_p)$ and will be reduced to Meijer's G-function if $e_j\ (j=1,\ldots\ldots,p)$, $f_i\ (i=1,\ldots,q)$ are positive integers, i. e.

$$H_{p,\,q}^{K,\,l}\left[x\left|\begin{matrix}\{(a_p, e_p)\}\\\{(b_q, f_q)\}\end{matrix}\right.\right]$$

$$=(2\pi)^D\prod_{j=1}^{q}f_j^{(b_j-\frac{1}{2})}\prod_{j=1}^{p}e_j^{(\frac{1}{2}-a_j)}\,G_{\sum_{j=1}^{p}e_j,\ \sum_{j=1}^{q}f_j}^{\sum_{j=1}^{K}f_j,\ \sum_{j=1}^{l}e_j}\left(\frac{\prod_{j=1}^{p}e_j^{e_j}}{\prod_{j=1}^{q}f_j^{f_j}}x\left|\begin{matrix}\triangle(e_1, a_1),\ldots,\triangle(e_p, a_p)\\\triangle(f_1, b_1),\ldots,\triangle(f_q, b_q)\end{matrix}\right.\right), \quad (2)$$

where $D=\sum_{j=1}^{K}\frac{1-f_j}{2}-\sum_{j=K+1}^{q}\frac{1-f_j}{2}+\sum_{j=1}^{l}\frac{1-e_j}{2}-\sum_{j=l+1}^{p}\frac{1-e_j}{2}$ and $\triangle(m, a)$ represents a set of m parameters $\frac{a}{m}, \frac{a+1}{m}, \ldots, \frac{a+m-1}{m}$.

According to Braaksma[6]

$$H\,{}^{K,\,l}_{p,\,q}\left[\,x\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\\{(b_q,f_q)\}\end{matrix}\right] = 0\,(|\,x\,|^{\alpha})\quad \text{for small } x,$$

where $\sum\limits_{j=1}^{p} e_j - \sum\limits_{j=1}^{q} f_j \leqslant 0$ and $\alpha = \min R\left(\dfrac{b_h}{f_h}\right)$, $(h = 1, \ldots\ldots, K)$

and

$$H\,{}^{K,\,l}_{p,\,q}\left[\,x\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\\{b_q,f_q)\}\end{matrix}\right] = 0\,(\,|\,x\,|^{\beta}\,)\quad \text{for large } x,$$

where $\sum\limits_{j=1}^{p} e_j - \sum\limits_{j=1}^{q} f_j < 0;\ \sum\limits_{j=1}^{l} e_j - \sum\limits_{j=l+1}^{p} e_j + \sum\limits_{j=1}^{K} f_j - \sum\limits_{j=K+1}^{q} f_j \equiv \lambda > 0,\ |\arg x| < \tfrac{1}{2}\pi\lambda$

and $\beta = \max R\left(\dfrac{a_j-1}{e_j}\right)$ $(j = 1, \ldots\ldots, l)$.

The following formulae will be used in the present work.

$$\int_0^t x^{\rho-1}(t-x)^{\beta-1}\, H\,{}^{K,\,l}_{p,\,q}\left[\,z\,x^{m/h}\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\\{(b_q,f_q)\}\end{matrix}\right]dx$$

$$= (2\pi)^{(1-h)A}\, h^{B}\, t^{\rho+\beta-1}\,\Gamma(\beta)\, H\,{}^{Kh,\,lh+1}_{ph+1,\,qh+1}\left[(z\,h^{\tau})^{h}\,t^{m}\,\middle|\,\begin{matrix}(1-\rho,m),\ \{(\triangle(h,a_p),e_p)\}\\\{(\triangle(h,b_q),f_q)\},\ \triangle(1-\rho-\beta,m)\end{matrix}\right], \qquad (3)$$

where m is a positive number and h is a positive integer,

$A = K + l - \dfrac{p}{2} - \dfrac{q}{2}$, $B = \sum\limits_{j=1}^{q} b_j - \sum\limits_{j=1}^{b} a_j + \dfrac{p}{2} - \dfrac{q}{2} + 1$, $\tau = \sum\limits_{j=1}^{p} e_j - \sum\limits_{j=1}^{q} f_j$, $\{(\triangle(t,\delta_r),\gamma_r)\}$ stands for

$\{(\dfrac{\delta_r}{t},\gamma_r)\},\ldots,\{(\dfrac{\delta_r+t-1}{t},\gamma_r)\}$ and provided that $R(\beta) > 0$, $R(\rho + \dfrac{mb_j}{hf_j}) > 0$ $(j = 1, \ldots, K)$,

$\sum\limits_{j=1}^{k} f_j - \sum\limits_{j=k+1}^{q} f_j + \sum\limits_{j=1}^{l} e_j - \sum\limits_{j=l+1}^{p} e_j \equiv M > 0$, $\tau \leqslant 0$ and $|\arg z| < \tfrac{1}{2}\pi M$

$$\int_0^t x^{\rho-1}(t-x)^{\beta-1}\, H\,{}^{K,\,l}_{p,\,q}\left[\,z\,x^{m/h}(t-x)^{n/h}\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\\{(b_q,f_q)\}\end{matrix}\right]dx$$

$$= (2\pi)^{(1-h)A}\, h^{B}\, t^{\rho+\beta-1}\, H\,{}^{Kh,\,lh+2}_{ph+2,\,qh+1}\left[(z\,h^{\tau})^{h}\,t^{m+n}\,\middle|\,\begin{matrix}(1-\rho,m),\ (1-\beta,n),\ \{(\triangle(h,a_p),e_p)\}\\\{(\triangle(h,b_q),f_q)\},\ (1-\rho-\beta,m+n)\end{matrix}\right], \qquad (4)$$

provided that m and n are positive numbers and h is a positive integer,

$R(\rho + \dfrac{mb_j}{hf_j}) > 0$, $R(\beta + \dfrac{nb_j}{hf_j}) > 0$ $(j = 1, \ldots\ldots, K)$, $M > 0, \tau \leqslant 0$ and $|\arg z| < \tfrac{1}{2}\pi M$.

$$\int_0^t x^{\rho-1}(t-x)^{\beta-1}\, H\,{}^{K,\,l}_{p,\,q}\left[z\,x^{m/h}(t-x)^{-n/h}\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\\{(b_q,f_q)\}\end{matrix}\right]dx = \theta,\ \phi \text{ or } \psi, \qquad (5)$$

According as $m > n$, $< n$ or $= n$, where

$$\theta = (2\pi)^{(1-h)A}\, h\, B\, t^{\rho+\beta-1}\; H^{Kh+1,\, lh+1}_{ph+1,\, qh+2}\left[(zh^{\tau})^h\, t^{m-n} \,\middle|\, \begin{matrix} (1-\rho, m), \{(\triangle(h, a_p), e_p)\} \\ (\beta, n), \{(\triangle(h, b_q), f_q)\}, (1-\rho-\beta, m-n) \end{matrix} \right],$$

$$\phi = (2\pi)^{(1-h)A}\, h^B\, t^{\rho+\beta-1}\; H^{Kh+1,\, lh+1}_{ph+2,\, qh+1}\left[\frac{(zh^{\tau})^h}{t^{n-m}} \,\middle|\, \begin{matrix} (1-\rho, m), \{(\triangle(h, a_p), e_p\}, (\rho+\beta, n-m) \\ (\beta, n), \{(\triangle(h, b_q), f_q)\} \end{matrix} \right]$$

and

$$\psi = \frac{(2\pi)^{(1-h)A}\, h^B t^{\rho+\beta-1}}{\Gamma(\rho+\beta)}\; H^{Kh+1,\, lh+1}_{ph+1,\, qh+1}\left[(zh^{\tau})^h \,\middle|\, \begin{matrix} (1-\rho, m), \{(\triangle(h, a_p), e_p)\} \\ (\beta, m), \{(\triangle(h, b_q), f_q)\} \end{matrix} \right],$$

where m, n are positive numbers and h is a positive integer and the condition of validity being

$R\{\beta - \frac{n}{he_j}(a_j - 1)\} > 0$ $(j = 1, \ldots\ldots, l)$, $R(\rho + \frac{mb_j}{hf_j} > 0$ $(j = 1, \ldots\ldots, K)$, $M > 0$, $\tau < 0$ and $|\arg z| < \frac{1}{2}\pi M$.

when $n=m$, n is to be replaced by m in the above conditions.

The formulae (3), (4) and (5) can easily be established by replacing the H-function (with $h=1$) on the left hand side by its equivalent contour integral as given in (1), changing the order of integrations which is justifiable due to the absolute convergence of the integrals, evaluating the inner integral[7] and applying the multiplication formula for H-function, viz.

$$H^{K,\, l}_{p,\, q}\left[z\, x^{m/h} \,\middle|\, \begin{matrix} \{(a_p, e_p)\} \\ \{(b_q, f_q)\} \end{matrix} \right] = (2\pi)^{(1-h)A}\, h^B\, H^{Kh,\, lh}_{ph,\, qh}\left[(z\, h^{\tau})^h x^m \,\middle|\, \begin{matrix} \{(\triangle(h, a_p), e_p)\} \\ \{(\triangle(h, b_q), f_q)\} \end{matrix} \right].$$

The finite difference operator E is[8]

$$E_a f(a) = f(a+1): \tag{6}$$

$$(\alpha\,;\, n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}: \tag{7}$$

We shall now obtain the following formulae :

$$\int_0^t x^{\rho-1}(t-x)^{\beta-1}\, {}_uF_v\left\{ \begin{matrix} \alpha_1, \ldots, \alpha_u \\ \beta_1, \ldots, \beta_v \end{matrix} ; C x^{\nu}(t-x)^{\mu} \right\} H^{K,\, l}_{p,\, q}\left[z x^{m/h} \,\middle|\, \begin{matrix} \{(a_p, e_p)\} \\ \{(p_q, f_q)\} \end{matrix} \right] dx$$

$$= (2\pi)^{(1-h)A}\, h^B\, t^{\rho+\beta-1} \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u} (\alpha_j;\, r)\, \Gamma(\beta + r\mu)\, C^r\, t^{r(\nu+\mu)}}{\prod_{j=1}^{v} (\beta_j;\, r)\, (r)!} \,.$$

$$\cdot\, H^{Kh,\, lh+1}_{ph+1,\, qh+1}\left[(z\, h^{\tau})^h t^m \,\middle|\, \begin{matrix} (1-\rho-r\nu, m), \{(\triangle(h\; a_p)\; e_p)\} \\ \{(\triangle(h, b_q), f_q)\}, (1-\rho-\beta-r\nu-r\mu, m) \end{matrix} \right], \tag{8}$$

where m is a positive number, h is a positive integer, ν and μ are non-negative integers. The above formula holds if $u \leqslant v$ (or $u = v+1$ and $|C| < 1$), no one of the denominator parameters $\beta_1, \ldots\ldots, \beta_v$ is zero or a negative integer, $R(\beta) > 0$, $R(\rho + \frac{mb_j}{hf_j}) > 0$ $(j = 1, \ldots\ldots, K)$, $M > 0$, $\tau \leqslant 0$ and $|\arg z| < \frac{1}{2}\pi M$.

$$\int_0^t x^{\rho-1}(t-x)^{\beta-1} {}_uF_v \left\{ \begin{matrix}\alpha_1, \ldots, \alpha_u; \\ \beta_1, \ldots, \beta_v;\end{matrix} Cx^{\nu}(t-x)^{\mu} \right\} H^{K,l}_{p,q}\left[z\, x^{m/h}(t-x)^{n/h} \left| \begin{matrix}\{(a_p, e_p)\} \\ \{(b_q, f_q)\}\end{matrix}\right.\right] dx$$

$$= (2\pi)^{(1-h)A} h^B t^{\rho+\beta-1} \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(\alpha_j; r)\; C^r\; t^{r(\nu+\mu)}}{\prod_{j=1}^{v}(\beta_j; r)\,(r)!} \cdot$$

$$\cdot H^{Kh,\, lh+2}_{ph+2,\, qh+1}\left[(zh^{\tau})^h\, t^{m+n} \left| \begin{matrix}(1-\rho-r\nu, m), (1-\beta-r\mu, n), \{(\triangle(h, a_p), e_p)\} \\ \{(\triangle(h, b_q), f_q)\}, (1-\rho-\beta-r\nu-r\mu, m+n)\end{matrix}\right.\right] \tag{9}$$

provided that m, n are positive numbers, h is a positive integer, ν and μ are non-negative integers, $u \leqslant v$ ($u = v+1$ and $|C| < 1$), no one of the denominator parameters $\beta_1, \ldots\ldots, \beta_v$ is zero or a negative integer, $R(\rho + \frac{mb_j}{hf_j}) > 0$, $R(\beta + \frac{nb_j}{hf_j}) > 0$ $(j = 1, \ldots, K)$, $M > 0$, $\tau \leqslant$ o and $|\arg z| < \frac{1}{2}\pi M$

$$\int_0^t x^{\rho-1}(t-x)^{\beta-1} {}_uF_v \left\{ \begin{matrix}\alpha_1, \ldots\ldots, \alpha_u \\ \beta_1, \ldots, \beta_v\end{matrix} Cx^{\nu}(t-x)^{\mu} \right\} H^{K,l}_{p,q}\left[zx^{m/h}(t-x)^{-n/h} \left| \begin{matrix}\{(a_p, e_p)\} \\ \{(b_q, f_q)\}\end{matrix}\right.\right] dx$$

$= \theta_1, \phi_1$ or ψ_1, according as $m > n, < n$, or $= n$, (10)

where

$$\theta_1 = (2\pi)^{(1-h)A} h^B t^{\rho+\beta-1} \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(\alpha_j; r)\, C^r\, t^{r(\nu+\mu)}}{\prod_{j=1}^{v}(\beta_j; r)\,(r)!} \cdot$$

$$\cdot H^{Kh+1,\, lh+1}_{ph+1,\, qh+2}\left[(zh^{\tau})^h\, t^{m-n} \left| \begin{matrix}(1-\rho-r\nu, m), \{(\triangle(h, a_p), e_p)\} \\ (\beta+r\mu, n), \{(\triangle(h, b_q), f_q)\}, (1-\rho-\beta-r\nu-r\mu, m-n)\end{matrix}\right.\right],$$

$$\phi_1 = (2\pi)^{(1-h)A} h^B t^{\rho+\beta-1} \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(\alpha_j; r)\, C^r\, t^{r(\nu+\mu)}}{\prod_{j=1}^{v}(\beta_j; r)\,(r)!} \cdot$$

$$\cdot H^{Kh+1,\, lh+1}_{ph+2,\, qh+1}\left[\frac{(zh^{\tau})^h}{t^{n-m}} \left| \begin{matrix}(1-\rho-r\nu, m), \{(\triangle(h, a_p), e_p)\}, (\rho+\beta+r\nu+r\mu, n-m) \\ (\beta+r\mu, n), \{(\triangle(h, b_q), f_q)\}\end{matrix}\right.\right]$$

and

$$\psi_1 = (2\pi)^{(1-h)A} h^B t^{\rho+\beta-1} \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(\alpha_j; r)\, C^r\, t^{r(\nu+\mu)}}{\prod_{j=1}^{v}(\beta_j; r)\,(r)!\,\Gamma(\rho+\beta+r\nu+r\mu)} \cdot$$

$$\cdot\, H^{Kh+1,\, lh+1}_{ph+1,\, qh+1}\left[(zh^{\tau})^{h}\,\middle|\,\begin{matrix}(1-\rho-r\nu,\, m),\ \{(\triangle(h, a_p), e_p)\}\\ (\beta+r\mu,\, m),\ \{(\triangle(h, b_q), f_q)\}\end{matrix}\right],$$

where m, n are positive numbers, h is a positive integer and ν and μ are non-negative integers. The formula (10) exists when $u \leqslant v$ ($u = v + 1$ and $|C| < 1$), no one of $\beta_1, \ldots\ldots\ldots, \beta_v$ is zero or a negative integer, $R\left(\rho + \frac{mb_j}{hf_j}\right) > 0$ $(j = 1, \ldots, K)$, $R\{\beta - \frac{n}{h e_j}(a_j - 1)\} > 0$ $(j = 1, \ldots, l)$, $M > 0$, $\tau < 0$ and $|\arg z| < \frac{1}{2} M\pi$ when $n = m$, n is to be replaced by m in the above conditions.

Proof of Integral (8)—

On multiplying both sides of (3) by $\dfrac{\prod_{j=1}^{u}\Gamma(\alpha_j+\delta)C^{\delta}}{\prod_{j=1}^{v}\Gamma(\beta_j+\delta)}$ and applying the operators $e^{E_\rho^\nu}E_\beta^\mu E_\delta$, we get

$$e^{E_\rho^\nu}\, E_\beta^\mu\, E_\delta\left[\int_0^t x^{\rho-1}(t-x)^{\beta-1} H^{K,\,l}_{p,\,q}\left[z\,x^{m/h}\,\middle|\,\begin{matrix}\{(a_p, e_p)\}\\ \{(b_q, f_q)\}\end{matrix}\right]\frac{\prod_{j=1}^{u}\Gamma(\alpha_j+\delta)\,C^{\delta}}{\prod_{j=1}^{v}\Gamma(\beta_j+\delta)}\,dx\right]$$

$$= e^{E_\rho^\nu}\, E_\beta^\mu\, E_\delta\left[\frac{\prod_{j=1}^{u}\Gamma(\alpha_j+\delta)\,(2\pi)^{(1-h)A}\,h^{B}\,t^{\rho+\beta-1}\,C^{\delta}\,\Gamma(\beta)}{\prod_{j=1}^{v}\Gamma(\beta_j+\delta)}\right.\,\cdot$$

$$\left.\cdot\, H^{Kh,\, lh+1}_{ph+1,\, qh+1}\left[(z\,h^{\tau})^{h}\,t^{m}\,\middle|\,\begin{matrix}(1-\rho,\, m),\ \{(\triangle(h, a_p), e_p)\}\\ \{(\triangle(h, b_q), f_q)\},\ (1-\rho-\beta,\, m)\end{matrix}\right]\right]. \tag{11}$$

Expanding both sides of (11) and using (6), we have

$$\sum_{r=0}^{\infty}\int_0^t x^{\rho-1}(t-x)^{\beta-1}\, H^{K,\,l}_{p,\,q}\left[z\,x^{m/h}\,\middle|\,\begin{matrix}\{(a_p, e_p)\}\\ \{(b_q, f_q)\}\end{matrix}\right]\frac{\prod_{j=1}^{u}\Gamma(\alpha_j+\delta+r)\,C^{\delta+r}\,x^{r\nu}\,(t-x)^{r\mu}}{\prod_{j=1}^{v}\Gamma(\beta_j+\delta+r)\,(r)!}\,dx$$

$$= (2\pi)^{(1-h)A}\,h^{B}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{u}\Gamma(\alpha_j+\delta+r)\,C^{\delta+r}\,t^{\rho+\beta+r\nu+r\mu-1}\,\Gamma(\beta+a\mu)}{\prod_{j=1}^{v}\Gamma(\beta_j+\delta+r)\,(r)!}\,\cdot$$

$$\cdot\, H^{Kh,\, lh+1}_{ph+1,\, qh+1}\left[(z\,h^{\tau})^{h}\,t^{m}\,\middle|\,\begin{matrix}(1-\rho-r\nu,\, m),\ \{(\triangle(h, a_p), e_p)\}\\ \{(\triangle(h, b_q), f_q)\},\ (1-\rho-\beta-r\nu-r\mu,\, m)\end{matrix}\right]. \tag{12}$$

Now using (7), changing the order of summation and integration of the left hand side and replacing $(\alpha_j + \delta)$ by α_j and $(\beta_j + \delta)$ by β_j we get (8). The change of order of summation and integration involved in the process can be easily justified[8].

The results (9) and (10) can be established exactly on the same lines using the same multiplyer and operators except that the formulae (4) and (5) are used respectively instead of (3).

PARTICULAR CASES

(i) Putting $\nu=0$ in (8), we have

$$\int_0^t x^{\rho-1}(t-x)^{\beta-1}\,{}_uF_v\left\{\begin{matrix}\alpha_1,\ldots,\alpha_u\\ \beta_1,\ldots,\beta_v\end{matrix};\,C(t-x)^{\mu}\right\} H^{K,l}_{p,q}\left[z\,x^{m/h}\,\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right]dx$$

$$=(2\pi)^{(1-h)A}\,h^B\,t^{\rho+\beta-1}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{u}(\alpha_j\,;r)\,\Gamma(\beta+r\mu)C^r\,t^{r\mu}}{\prod_{j=1}^{v}(\beta_j;r)\,(r)\,!}\,.$$

$$.\,H^{Kh,\,lh+1}_{ph+1,\,qh+1}\left[(z\,h^{\tau})^h\,t^m\,\middle|\,\begin{matrix}(1-\rho,m),\,\{(\triangle\,(h,a_p),e_p)\}\\ \{(\triangle\,(h,b_q),f_q)\},\,(1-\rho-\beta-r\mu,m)\end{matrix}\right], \qquad (13)$$

The conditions of validity being the same as given for (8).

(ii) Taking $C=t=v=\mu=1$, $u=2$, $\beta_1=\beta$, $\nu=0$ in (8), replacing the H-function on the right hand side by its equivalent contour integral as given in (1), then changing the order of summation and integration and evaluating the inner summation with the help of Gauss's theorem[7], we have

$$\int_0^1 x^{\rho-1}(1-x)^{\beta-1}\,{}_2F_1\left(\begin{matrix}\alpha_1,\alpha_2\\ \beta\end{matrix};\,1-x\right) H^{K,l}_{p,q}\left[z\,x^{m/h}\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right]dx$$

$$=(2\pi)^{(1-h)A}\,h^B\,\Gamma(\beta)\,H^{Kh,\,lh+2}_{ph+2,\,qh+2}\left[(z\,h^{\tau})^h\,\middle|\,\begin{matrix}(1-\rho,m),\,(1+\alpha_1+\alpha_2-\rho-\beta,m),\,\{(\triangle\,(h,a_p),e_p)\}\\ \{(\triangle\,(h,b_q),f_q)\},\,(1+\alpha_1-\rho-\beta,m),\,(1+\alpha_2-\rho-\beta,m)\end{matrix}\right], \qquad (14)$$

provided that m is positive number, h is a positive integer, $R\,(\rho+\beta-\alpha_1-\alpha_2)>0$, $R\,(\beta)>0$ $R\left(\rho+\dfrac{mb_j}{hf_j}\right)>0\ (j=1,\ldots,K)$, $M>0$, $\tau\leqslant 0$ and $|\arg z|<\frac{1}{2}\pi M$.

(iii) In (8) on using the formula (2), we get a known result recently obtained by Chhabra & Singh[2] Further on taking $C=\nu=h=t=v=1$, $u=2$, $\mu=0$, replacing β by $\beta-\gamma-n+1$, α_1 by $-n$, where n is a positive integer, α_2 by β, β_1 by γ, H-function on the right hand side by its equivalent contour integral, then changing the order of summation and integration and using Saalschutz's theorem[7], we obtain result due to Sharma[4].

(iv) Taking $C=t=\nu=v=1$, $u=2$, $\mu=0$ in (8) and rearranging the parameters in H-function on the right hand side, we get a result given by Sharma[3].

(v) On taking $C=t=\nu=h=v=1$, $u=2$, $\alpha_1=\alpha$, $\alpha_2=2\rho$, $\beta_1=\rho+\dfrac{\alpha}{2}+\dfrac{1}{2}$, $n=m$, $\mu=0$ in (9), replacing β by ρ and H-function on the right hand side by its equivalent contour integral as given in (1) changing the order of summation and integration and applying the Watson's theorem[9], we obtain

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\,{}_2F_1\left(\begin{matrix}\alpha,2\rho\\ \rho+\frac{\alpha}{2}+\frac{1}{2}\end{matrix};x\right) H^{K,l}_{p,q}\left[z\,x^m(1-x)^m\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right]dx$$

$$=\frac{\sqrt{\pi}\,\Gamma(\rho+\frac{\alpha}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{\alpha}{2})\,\Gamma(\frac{1}{2}+\rho)}\,H^{K,\,l+4}_{p+4,\,q+3}\left[z\,\middle|\,\begin{matrix}(1-\rho,m),\,(1-\rho,m),\,(\frac{1}{2}-\rho,m),\,(1+\frac{\alpha}{2},m),\,\{(a_p,e_p)\}\\ \{(b_q,f_q)\},\,(1-2\rho,2m),\,(\frac{1}{2}+\frac{\alpha}{2}-\rho,m),\,(\frac{1}{2},m)\end{matrix}\right] \qquad (15)$$

where m is a positive number, $R(\alpha) < 1$, $R\left(\rho + \frac{mb_j}{f_j}\right) > 0$ $(j = 1, \ldots, K)$, $M > 0$, $\tau \leq 0$ and $|\arg z| < \frac{1}{2}\pi M$.

(vi) Setting $C = t = h = \nu = v = 1$, $u = 2$, $\mu = 0$ and rearranging the parameters in H-function on the right hand side, (9) yields a known result due to Sharma[3]. Further using (2) we get another result given recently by Bajpai[1].

(vii) If we take $C = t = \nu = h = 1$, $\mu = 0$, $u = 4$, $v = 3$, $\alpha_1 = \alpha$, $\alpha_2 = 1 + \frac{\alpha}{2}$, $\alpha_3 = 1 + \alpha - \rho$, $\alpha_4 = \nu$, $\beta_1 = \frac{\alpha}{2}$, $\beta_2 = \rho$ and $\beta_3 = 1 + \alpha - \nu$ in (10) (with $m < n$), replace β by $1 + \alpha - 2\rho$, n by $2m$, H-function on the right hand side by its equivalent contour integral as given in (1), change the order of summation and integration and evaluate the inner summation with the help of Dougall's second theorem[9], we get

$$\int_0^1 x^{\rho-1}(1-x)^{\alpha-2\rho}\ {}_4F_3\left(\begin{matrix}\alpha, 1+\frac{\alpha}{2}, 1+\alpha-\rho, \nu \\ \frac{\alpha}{2}, \rho, 1+\alpha-\nu\end{matrix}; x\right) H^{K,l}_{p,q}\left[z x^m (1-x)^{-2m} \middle| \begin{matrix}\{(a_p, e_p)\} \\ \{(b_q, f_q)\}\end{matrix}\right] dx$$

$$= \frac{\Gamma(\rho)\,\Gamma(1+\alpha-\nu)}{\Gamma(1+\alpha)\,\Gamma(\rho-\nu)} H^{K+2,\, l+1}_{p+3,\, q+2}\left[z \middle| \begin{matrix}(1-\rho, m), \{(a_p, e_p)\}, (\rho, m), (1+\alpha-\nu-\rho, m) \\ (1+\alpha-2\rho, 2m), (-\nu, m), \{(b_q, f_q)\}\end{matrix}\right], \tag{16}$$

provided m is a positive number, $R(\nu) < 0$, $R\left(\rho + \frac{mb_j}{f_j}\right) > 0$ $(j = 1, \ldots, K)$,

$R\left\{1 + \alpha - 2\rho - 2m\left(\frac{a_j - 1}{e_j}\right)\right\} > 0$ $(j = 1, \ldots, l)$, $M > 0$, $\tau < 0$ and $|\arg z| < \frac{1}{2}\pi M$.

(viii) On setting $C = t = \nu = v = 1$, $u = 2$, $\mu = 0$ and rearranging the parameters in H-function on the right hand side, (10) (with $n = m$) yields a result obtained by Sharma[3]. Further if we apply the formula (2), we get a next result given by Bajpai[1].

(ix) Lastly with $C = t = \nu = v = h = 1$, $u = 2$, $\mu = 0$, replacing β by $\beta - \rho$, α_1 by α, α_2 by β, β_1 by γ, proceeding on the same lines as in (*vii*) and evaluating the inner summation with the help of Gauss's theorem[7], (10) (with $n = m$) yields

$$\int_0^1 x^{\rho-1}(1-x)^{\beta-\rho-1}\ {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}; x\right) H^{K,l}_{p,q}\left[z x^m (1-x)^{-m} \middle| \begin{matrix}\{(a_p, e_p)\} \\ \{(b_q, f_q)\}\end{matrix}\right] dx$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)} H^{K+2,\, l+1}_{p+2,\, q+2}\left[z \middle| \begin{matrix}(1-\rho, m), \{(a_p, e_p)\}, (\gamma-\rho, m) \\ (\beta-\rho, m), (\gamma-\alpha-\rho, m), \{(b_q, f_q)\}\end{matrix}\right], \tag{17}$$

where m is a positive number, $R(\gamma - \alpha - \rho) > 0$, $R\left(\rho + \frac{mb_j}{f_j}\right) > 0$ $(j = 1, \ldots, K)$,

$R\left\{\beta - \rho - m\left(\frac{a_j - 1}{e_j}\right)\right\} > 0$ $(j = 1, \ldots, l)$, $\tau < 0$, $M > 0$ and $|\arg z| < \frac{1}{2}\pi M$.

If we use the formula (2), (17) yields a result obtained by Sharma[4]. The results similar to (13) can also be obtained from (9) and (10) on taking $\nu = 0$

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