REISSNER-SAGOCI PROBLEM FOR A NON-HOMOGENEOUS SOLID

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(*Received* 21 *June*, 1971)

A solution of the problem is given when a circular part of non-homogeneous semi-infinite medium is subject to axisymmetric twisting deformation. The results for the homogeneous isotropic case are a particular case of the problem.

The standard Reissner Sagoci problem[1] is that of determining the components of stress and displacement in the interior of a semi-infinite elastic solid $z \geqslant 0$ when a circular area $0 \leqslant r \leqslant a$ of the boundary surface $z = 0$ is forced to rotate through an angle γ about an axis normal to the undeformed plane surface of the solid. It is assumed that the part of the boundary surface lying outside the circle is stress free. Reissner & Sagoci[1] employed a system of oblate spheroidal coordinates. The same problem was also approached by Sneddon[2,3] via the Hankel Transforms. The main object of this paper is to generalize the standard Reissner-Sagoci problem by taking a non-homogeneous semi-infinite medium. Due to the medium being non-homogeneous, we take shear modulus as $\frac{\mu_0}{r^\alpha}$, where μ_0 is a constant and $\alpha \leqslant 0$. We reduce the problem to dual integral equations. These integral equations are solved by the method based upon the work of Copson[4] (which is an extension of the work by Sneddon[5]. This enables us to derive the simple expressions for the physical quantities.

PROBLEM, FUNDAMENTAL EQUATION AND BOUNDARY CONDITIONS

We use cylindrical coordinates (r, θ, z), the displacement vector has only one non-vanishing component $u_\theta(r, z)$ and the stress tensor has only two non-vanishing components $\sigma_{r\theta}(r, z)$ and $\sigma_{\theta z}(r, z)$. The stress strain relations reduce to simple equations.

$$\sigma_{r\theta}(r, z) = \mu \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \tag{1}$$

$$\sigma_{\theta z}(r, z) = \mu \frac{\partial u_\theta}{\partial z} \tag{2}$$

We suppose that the rigidity of the solid is given by

$$\mu = \frac{\mu_0}{r^\alpha}, \ \alpha \leqslant 0 , \tag{3}$$

where μ_0 and α are constants. Then,

$$\sigma_{r\theta}(r, z) = \frac{\mu_0}{r^\alpha} \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \tag{4}$$

$$\sigma_{\theta z}(r, z) = \frac{\mu_0}{r^\alpha} \frac{\partial u_\theta}{\partial z} \tag{5}$$

Two equations of equilibrium are satisfied identically and the remaining one reduces to

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{(1-\alpha)}{r} \frac{\partial u_\theta}{\partial r} + \frac{(\alpha - 1)}{r^2} u_\theta + \frac{\partial^2 u_\theta}{\partial z^2} = 0 \tag{6}$$

When a circular part $(0 \leqslant r < a, z = 0)$ of a non-homogeneous semi-infinite solid is forced to rotate with prescribed tangential displacement, it is assumed that the part of the boundary surface which lies outside the circular part is stress free. Due to these facts we have following boundary conditions :

$$u_\theta(r, 0) = f(r) \qquad 0 \leqslant r < a \tag{7}$$

$$\sigma_{\theta z}(r, 0) = 0 \qquad r > a. \tag{8}$$

$\sigma_{\theta z}, u_\theta, \sigma_{\theta r}$, all tend to zero as $(r^2 + z^2) \to \infty$, and $f(r)$ is prescribed function.

SOLUTION OF THE PROBLEM

Now we take the suitable solution of (6) in the form

$$u_\theta (r, z) = r^{\alpha/2}\, Hv\, [\, \xi^{-1} \psi (\xi)\, e^{-\xi z},\ \xi \to r\,] \tag{9}$$

where

$$v = \frac{1}{2} (2 - \alpha)\,, \tag{10}$$

and the operator Hv is defined by

$$Hv\, [\, \psi (\xi, z),\ \xi \to r\,] = \int_0^\infty \xi \psi (\xi, z)\, Jv\, (\xi r)\, d\xi \tag{11}$$

We have

$$\sigma_{\theta z} (r, z) = \frac{\mu_0}{r^{\alpha/2}}\, Hv\, [\, \psi (\xi)\, e^{-\xi z},\ \xi \to r\,] \tag{12}$$

The boundary conditions (7), and (8) reduce to dual integral equations

$$Hv\, [\, \xi^{-1} \psi (\xi),\ \xi \to r\,] = \frac{f(r)}{r^{\alpha/2}}, \qquad 0 < r < a, \tag{13}$$

$$Hv\, [\, \psi (\xi),\ \xi \to r\,] = 0, \qquad r > a \tag{14}$$

We shall express $\psi (\xi)$ in terms of an unknown function $g(t)$ through the equation

$$\psi (\xi) = \left(\frac{\pi \xi}{2} \right)^{\frac{1}{2}} \int_0^a t^{3/2 - v}\, g(t)\, J_{v-1/2} (\xi t)\, dt \tag{15}$$

Now

$$Hv\, [\, \psi (\xi),\ \xi \to r\,] = \sqrt{\frac{\pi}{2}} \int_0^a g(t)\, t^{3/2 - v}\, dt \int_0^\infty \xi^{3/2}\, Jv\, (\xi r) \cdot J_{v-1/2} (\xi t)\, d\xi$$

$$= - r^{v-1} \sqrt{\frac{\pi}{2}} \int_0^a g(t)\, t^{3/2 - v}\, dt\, \frac{\partial}{\partial r}\, r^{1-v} \int_0^\infty \xi^{1/2}\, J_{v-1} (\xi r) \cdot J_{v-1/2} (\xi t)\, d\xi \tag{16}$$

and making use of the integral

$$\int_0^\infty \xi^{1/2}\, J_{v - \frac{1}{2}} (\xi t)\, J_{v-1} (\xi r)\, d\xi = \sqrt{\frac{2}{\pi}}\, t^{1/2 - v}\, r^{v-1}\, (t^2 - r^2)^{-1/2}\, H(t - r) \tag{17}$$

where $H(t - r)$ denotes Heavy-side's unit function (which can be easily obtained from Erdelyi[6]), we see that the equation (14) is automatically satisfied, whatever may be the form of $g(t)$. On using (17), we can easily get from (16) the following :

$$Hv\, [\, \psi (\xi),\ \xi \to r\,] = - r^{v-1} \frac{\partial}{\partial r} \int_r^a \frac{t^{2-2v}\, g(t)\, dt}{(t^2 - r^2)^{\frac{1}{2}}} \qquad 0 < r < a \tag{18}$$

Putting the value of $\psi(\xi)$ from (15) in (13) and making use of integral[6], we obtain

$$\int_0^\infty (\xi)^{\frac{1}{2}} J_v(\xi r)\, J_{v-1/2}(\xi t)\, d\xi = \sqrt{\frac{2}{\pi}}\; r^{-v} t^{v-\frac{1}{2}} (r^2-t^2)^{-1/2} H(r-t) \tag{19}$$

we get

$$\int_0^r \frac{t\, g(t)\, dt}{(r^2-t^2)^{\frac{1}{2}}} = f(r)\, r^{v-\alpha/2} \tag{20}$$

The equation (20) is Abel type, which is easily solved to give

$$g(t) = \frac{2}{\pi}\int_0^t \frac{\partial}{\partial r}\left\{ f(r).\, r^{v-\alpha/2}\right\} .\ \frac{dr}{(t^2-r^2)^{\frac{1}{2}}} \tag{21}$$

We can easily show

$$H_v[\xi^{-1}\psi(\xi),\ \xi \to r] = r^{-v}\int_0^a \frac{t\, g(t)\, dt}{(r^2-t^2)^{\frac{1}{2}}} \qquad a < r \tag{22}$$

Using (12) and (18) we get

$$\sigma_{\theta z}(r, 0) = \mu_0\, r^{v-\alpha/2-1}\frac{\partial}{\partial r}\int_r^a \frac{t^{2-2v}\, g(t)\, dt}{(t^2-r^2)^{\frac{1}{2}}} \qquad a > r \tag{23}$$

Similarly, from the equations (9) and (22), we deduce

$$(u_\theta)_{z=0} = r^{\alpha/2-v}\int_0^a \frac{t\, g(t)\, dt}{(r^2-t^2)^{\frac{1}{2}}} \qquad a < r \tag{24}$$

The torque T required to produce the rotation is

$$T = -2\pi\int_0^a r^2\, (\sigma_{\theta z})_{z=0}\, dr \tag{25}$$

Substituting the value $(\sigma_{\theta z})_{z=0}$ from (23) in (25) and integrating by parts and then changing the order of integrations, we get

$$T = \frac{\mu_0\, (\pi)^{3/2}\, (v+1-\alpha/2)\, \Gamma\left(\dfrac{v-\alpha/2+1}{2}\right)}{\Gamma\left(\dfrac{v-\alpha/2+2}{2}\right)}\int_0^a t^{2-v-\alpha/2}\, g(t)\, dt \tag{26}$$

The solution of (20) can be written as

$$g(t) = \frac{2}{\pi t}\,\frac{d}{dt}\int_0^t \frac{f(r)\, r^{v-\alpha/2+1}\, dr}{(t^2-r^2)^{\frac{1}{2}}} \tag{27}$$

Substituting (27) in (26) and integrating by parts, we get

$$T = \frac{2\sqrt{\pi}\, (v+1-\alpha/2)\, \Gamma\left(\dfrac{v-\alpha/2+1}{2}\right)\mu_0}{\Gamma\left(\dfrac{v-\alpha/2+2}{2}\right)} . \left[a^{1-v-\alpha/2}\int_0^a \frac{f(r)\, r^{v-\alpha/2+1}\, dr}{(a^2-r^2)^{\frac{1}{2}}} - \right.$$

$$\left. -\left(1-v-\frac{\alpha}{2}\right)\int_0^a t^{-v-\alpha/2}\, dt\int_0^t \frac{f(r)\, r^{v-\alpha/2+1}\, dr}{(t^2-r^2)^{\frac{1}{2}}}\right] \tag{28}$$

If we take $\alpha = 0$, then $v = 1$, we get the expression for torque in the homogeneous case

$$T = 8\,\mu_0 \int_0^a \frac{f(r)\, r^2\, dr}{(a^2 - r^2)^{\frac{1}{2}}} \tag{29}$$

The above expression is the same as derived by Sneddon[2]. The expression (28) can be written in terms of incomplete beta function.

$$T = \frac{2(\pi)^{\frac{1}{2}}(v+1-\alpha/2)\,\Gamma\left(\frac{v-\alpha/2+1}{2}\right)\mu_0}{\Gamma\left(\frac{v-\alpha/2+2}{2}\right)} \cdot \left[a^{1-v-\alpha/2} \int_0^a \frac{f(r)\, r^{v-\alpha/2+1}\, dr}{(a^2-r^2)^{\frac{1}{2}}} - \right.$$

$$\left. - \frac{(1-v-\alpha/2)}{2} \int_0^a f(r)\, r^{1-\alpha}\, B_{1-r^2/a^2}\left(\frac{1}{2}, \frac{v}{2} + \frac{\alpha}{4}\right) dr \right], \tag{30}$$

where

$$B_x(p, q) = \int_0^x u^{p-1}(1-u)^{q-1}\, du \qquad \mathrm{Re}[p] > 0, \qquad \mathrm{Re}[q] > 0 \tag{31}$$

$B_x(p, q)$ is incomplete beta function.

We have

$$\sigma_{r\theta}(r, \theta) = \frac{\mu_0}{r^{\alpha-1}} \frac{\partial}{\partial r}\left[r^{\alpha/2-1} \int_0^\infty A(\xi)\, J_v(\xi r)\, d\xi \right]$$

$$= \sqrt{\frac{\pi}{2}}\, \frac{\mu_0}{r^{\alpha-1}} \frac{\partial}{\partial r}\left[r^{\alpha/2-1} \int_0^a t^{3/2-v} g(t)\, dt \right.\,.$$

$$\left. \cdot \int_0^\infty (\xi)^{1/2} J_{v-1/2}(\xi t)\, J_v(\xi r)\, d\xi \right] \tag{32}$$

Making the use of (19), we can write (32) in the form

$$\sigma_{r\theta}(r, 0) = \frac{\mu_0}{r^{\alpha-1}} \frac{\partial}{\partial r}\left[\frac{1}{r^{v+1-\alpha/2}} \int_0^r \frac{t g(t)\, dt}{(r^2-t^2)^{\frac{1}{2}}} \right] \qquad 0 < r < a \tag{33}$$

$$= \frac{\mu_0}{r^{\alpha-1}} \frac{\partial}{\partial r}\left[\frac{1}{r^{v+1-\alpha/2}} \int_0^a \frac{t g(t)\, dt}{(r^2-t^2)^{\frac{1}{2}}} \right. \qquad a < r.$$

The value of $g(t)$ is known from (21).

PARTICULAR CASE

To illustrate the use of these formulae we consider the special case in which $f(r) = \gamma r$. It is easily shown from (21) that

$$g(t) = \frac{\gamma}{\sqrt{\pi}} \frac{(v-\alpha/2+1)\, t^{v-\alpha/2}\, \Gamma\left(\frac{v-\alpha/2+1}{2}\right)}{\Gamma\left(\frac{v-\alpha/2+2}{2}\right)} \qquad (v-\alpha/2+1) > 0 \tag{34}$$

Using (24) and (34), we have

$$(u_\theta)_{z=0} = \frac{\gamma\,(v-\alpha/2+1)\,\Gamma\left(\dfrac{v-\alpha/2+1}{2}\right) r^{\alpha/2-v}}{\sqrt{\ }\;\Gamma\left(\dfrac{v-\alpha/2+2}{2}\right)} \int_0^a \frac{t^{v-\alpha/2+1}\,dt}{(r^2-t^2)^{\frac{1}{2}}} \qquad a<r \tag{35}$$

(35) can be written in terms of incomplete beta function as :

$$(u_\theta)_{z=0} = \frac{\gamma\,(v-\alpha/2+1)\,\Gamma\left(\dfrac{v-\alpha/2+1}{2}\right) r}{2\sqrt{\pi}\;\Gamma\left(\dfrac{v-\alpha/2+2}{2}\right)}\, B_{a^2/r^2}\left(\frac{v}{2}-\frac{\alpha}{4}+1\;,\;\frac{1}{2}\right)$$

$$a<r, \qquad (v-\alpha/2+1)>0 \tag{36}$$

From (34) and (23) we find that

$$(\sigma_{\theta z})_{z=0} = \frac{\gamma\mu_0\,(v-\alpha/2+1)\,\Gamma\left(\dfrac{v-\alpha/2+1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\dfrac{v-\alpha/2+2}{2}\right)}\, r^{v-1-\alpha/2}\cdot\frac{\partial}{\partial r}\int_r^a \frac{t^{2-v-\alpha/2}\,dt}{(t^2-r^2)^{\frac{1}{2}}}, \qquad r<a \tag{37}$$

The above expression can be written in the form:

$$(\sigma_{\theta z})_{z=0} = \frac{\gamma\mu_0\,(v-\alpha/2+1)\,\Gamma\left(\dfrac{v-\alpha/2+1}{2}\right) r^{v-\alpha/2}}{\sqrt{\pi}\;\Gamma\left(\dfrac{v-\alpha/2+2}{2}\right)}\;.$$

$$\cdot\left[-\frac{a^{1-v-\alpha/2}}{(a^2-r^2)^{\frac{1}{2}}}+(1-v-\alpha/2)\int_r^a\frac{t^{-v-\alpha/2}\,dt}{(t^2-r^2)^{\frac{1}{2}}}\right], \qquad r<a \tag{38}$$

(38) can be written in terms of incomplete beta function as

$$(\sigma_{\theta z})_{z=0} = \frac{\gamma\mu_0\,(v-\alpha/2+1)\,\Gamma\left(\dfrac{v-\alpha/2+1}{2}\right) r^{v-\alpha/2}}{\sqrt{\pi}\;\Gamma\left(\dfrac{v-\alpha/2+2}{2}\right)}\;.$$

$$\cdot\left[-\frac{a^{1-v-\alpha/2}}{(a^2-r^2)^{\frac{1}{2}}}+\frac{(1-v-\alpha/2)}{2r^{v+\alpha/2}}\,B_{1-r^2/a^2}\left(\frac{1}{2},\,\frac{v}{2}+\frac{\alpha}{4}\right)\right]$$

$$(v-\alpha/2+1)>0, \tag{39}$$

If we take $\alpha=0$, $v=1$ then from (38) we find that

$$(\sigma_{\theta z})_{z=0} = \frac{-4\,\mu_0\,\gamma r}{\pi\,(a^2-r^2)^{\frac{1}{2}}} \tag{40}$$

Clearly (40) is in agreement with Sneddon[2]

From (34) and (26), we have

$$T = \frac{\gamma\mu_0\,\pi\left[\left(v+1-\dfrac{\alpha}{2}\right)\right]^2\left[\Gamma\left(\dfrac{v-\alpha/2+1}{2}\right)\right]^2}{\left[\Gamma\left(\dfrac{v-\alpha/2+2}{2}\right)\right]^2}\,\frac{a^{3-\alpha}}{3-\alpha}$$

$$(v-\alpha/2+1)>0 \tag{41}$$

Using (33) and (34), we can easily show that

$$(\sigma_{r\theta})_{z=0} = 0 \qquad 0 < r < a$$

$$= \frac{\mu_0 \gamma (v - \alpha/2 + 1) \Gamma\left(\frac{v - \alpha/2 + 1}{2}\right)}{\sqrt{\pi}\, r^{\alpha - 1} \Gamma\left(\frac{v - \alpha/2 + 1}{2}\right)} \cdot \left[-(v + 1 - \alpha/2) r^{-v-2+\alpha/2} \int_0^a \frac{t^{v-\alpha/2+1} dt}{(r^2 - t^2)^{\frac{1}{2}}} - \right.$$

$$\left. - r^{-v+\alpha/2} \int_0^a \frac{t^{v-\alpha/2+1}\, dt}{(r^2 - t^2)^{3/2}} \right], \qquad r > a\, ;\, (v - \alpha/2 + 1) > 0 \qquad (42)$$

We find that

$$\int_0^a \frac{t^{v-\alpha/2+1}\, dt}{(r^2 - t^2)^{3/2}} = \frac{a^{v-\alpha/2}}{(r^2 - a^2)^{\frac{1}{2}}} - \left(v - \frac{\alpha}{2}\right) \int_0^a \frac{t^{v-\alpha/2-1}\, dt}{(r^2 - t^2)^{\frac{1}{2}}} \qquad (43)$$

With the help of (42) and (43) we get

$$(\sigma_{r\theta})_{z=0} = \frac{\mu_0 \gamma (v - \alpha/2 + 1) \Gamma\left(\frac{v - \alpha/2 + 1}{2}\right)}{\sqrt{\pi}\, r^{\alpha - 1} \Gamma\left(\frac{v - \alpha/2 + 2}{2}\right)} \cdot \left[-(v + 1 - \alpha/2) r^{-v-2+\alpha/2} \int_0^a \frac{t^{v-\alpha/2+1} dt}{(r^2 - t^2)^{\frac{1}{2}}} - \right.$$

$$\left. - r^{-v+\alpha/2} \left\{ \frac{a^{v+\alpha/2}}{(r^2 - a^2)^{\frac{1}{2}}} - (v - \alpha/2) \int_0^a \frac{t^{v-\alpha/2-1}\, dt}{(r^2 - t^2)^{\frac{1}{2}}} \right\} \right],$$

$$r > a\, ;\, (v - \alpha/2 + 1) > 0 \qquad (44)$$

(44) can be written in terms of incomplete beta function as

$$(\sigma_{r\theta})_{z=0} = \frac{\mu_0 \gamma (v - \alpha/2 + 1) \Gamma\left(\frac{v - \alpha/2 + 1}{2}\right)}{\sqrt{\pi}\, r^{\alpha - 1} \Gamma\left(\frac{v - \alpha/2 + 2}{2}\right)} \cdot \left[- \frac{(v + 1 - \alpha/2)}{2r} B_{a^2/r^2}\left(1 + \frac{v}{2} - \frac{\alpha}{4}, \frac{1}{2}\right) - \right.$$

$$\left. - \frac{r^{-v+\alpha/2}\, a^{v-\alpha/2}}{(r^2 - a^2)^{\frac{1}{2}}} + \left(\frac{v - \alpha/2}{2r}\right) B_{a^2/r^2}\left(1 + \frac{v}{2} - \frac{\alpha}{4}, \frac{1}{2}\right) \right],$$

$$(v - \alpha/2 + 1) > 0\, ;\, r > a \qquad (45)$$

At the end, I have pointed out that if we take $\alpha = 0$, we get the results of the Reissner-Sagoci problem for a homogeneous material. All these results[2] are a particular case of this problem.

ACKNOWLEDGEMENTS

I wish to express my indebtedness to Dr. P. M. Gupta for his kind guidance during the preparation of this paper and to Dr. R. D. Agarwal for some helpful discussions.

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