STRUCTURE OF A PLANE BOUNDARY SHOCK WAVE

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Assuming the occurrence of a boundary shock wave as postulated by Martin, this paper studies its structure in conducting gas in the presence of a magnetic field.

Martin[1] postulated the concept of a boundary shock wave, a thin viscous region dominated by viscous compressive stress (rather than viscous shear) and the associated heat conduction in the gas adjacent to the body surface. He discussed the occurrence of boundary shock waves, developed the general theory for one dimensional laminar steady flow through a boundary shock in a perfect gas with longitudinal Prandtl number equal to unity. He also obtained explicit expressions for the jump conditions, the heat conduction co-efficient at the wall and the structure or variation of quantities through the boundary shock. From arguments similar to those given by Martin, Verma & Prasad[2], presumed the occurrence of boundary shock wave in a conducting gas in the presence of a magnetic field and extended the results of Martin to obtain the analogues of the well known Rankine Hugoniot jump relations, Prandtl number as well as other properties of the boundary shock. In this note, our purpose is to obtain the structure of a boundary shock wave in a conducting gas.

BASIC EQUATIONS

The equations for conservation of mass, momentum, field and energy in one-dimensional steady flow in a non-accelerating co-ordinate system are

$$\frac{d}{dx}\left(\rho u\right) = 0 \tag{1}$$

$$\frac{d}{dx}\left(\rho u^2\right) = \frac{df}{dx} \tag{2}$$

$$\frac{d}{dx}\left(u H\right) = 0 \tag{3}$$

and

$$\rho u \frac{d}{dx}\left(e + \frac{1}{2} u^2\right) = \frac{d}{dx}\left(-q + uf\right) \tag{4}$$

where x is the distance to the right of the boundary, e is the internal energy per unit mass, H is the intensity of the magnetic field, and f is the sum of the surface forces in the x-direction on an element of mass and is given by,

$$f = -p - \frac{\mu_e H^2}{8\pi} + \tau \tag{5}$$

$$\tau = \bar{\mu} \frac{du}{dx}, \left(\bar{\mu} = \frac{4}{3}\mu\right) \tag{6}$$

where μ is the shear viscosity coefficient and μ_e is the magnetic permeability. Also,

$$q = -k \frac{dT}{dx} \tag{7}$$

k being the coefficient of thermal conductivity.

The boundary conditions at $x = x_b = 0^+$, are,

$$\left.\begin{array}{c} u = u_b , T = T_b , q = q_b , \rho = \rho_b , \tau = \tau_b , \\ p = p_b , H = H_b \end{array}\right\} \tag{8}$$

and at $x \longrightarrow \infty$,

$$\frac{du}{dx} \longrightarrow 0 \, , \quad \frac{dT}{dx} \longrightarrow 0 \, . \tag{9}$$

On integration the equations (1) to (4) give

$$\rho \, u = \text{Constant.} \tag{10}$$

$$\rho \, u^2 + p + \frac{\mu_e \, H^2}{8 \, \pi} - \tau = \text{Constant.} \tag{11}$$

and

$$u \, H = \text{Constant.} \tag{12}$$

$$\rho \, u \, (e + \tfrac{1}{2} \, u^2) + q - u \, f = \text{Constant.} \tag{13}$$

STRUCTURE OF A BOUNDARY SHOCK

The differential equations (1) to (4) and the boundary conditions (8) and (9) may be put in terms of two variables u and T. Then it is convenient to make the equations dimensionless by defining and substituting the dependent variables,

$$\bar{u} = \frac{u}{u_b} \, , \; \overline{T} = \frac{T}{T_b} \, , \; \overline{H} = \frac{H}{H_b} \, . \tag{14}$$

and a new independent variable

$$\xi = \rho_b \, u_b \int \frac{du}{\widetilde{\mu}} \tag{15}$$

With the assumption, $\widetilde{P}_r$ = constant, $\widetilde{\mu}$ is proportional to k, since c_p is constant. Then from (11) and (13), we have

$$\frac{d\bar{u}}{d\xi} - \bar{u} - \frac{1}{\gamma M_b^2} \frac{\overline{T}}{\bar{u}} - \frac{\overline{H}^2}{2 M^2 H_b} = \frac{\tau_b}{\rho_b \, u_b} - 1 - \frac{1}{\gamma M_b^2} - \frac{1}{2 \, M^2 H_b} \tag{16}$$

and

$$\begin{aligned} \frac{d}{d\xi} \left[\frac{1}{\widetilde{P}_r} \left(\frac{\overline{T}}{(\gamma - 1) \, M_b^2} + \frac{\overline{T}}{2 \, M^2 H_b} \right) + \tfrac{1}{2} \, \bar{u}^2 \right] - \left[\frac{\overline{T}}{(\gamma - 1) \, M_b^2} + \frac{\overline{T}}{2 \, M^2 H_b} + \tfrac{1}{2} \, \bar{u}^2 \right] \\ = - \left[\frac{1}{(\gamma - 1) \, M_b^2} + \frac{1}{2 \, M^2 H_b} + \frac{1}{2} \right] + \left[\frac{- q_b}{\rho_b \, u_b^3} + \frac{\tau_b}{\rho_b \, u_b^2} \right] \end{aligned} \tag{17}$$

and the boundary conditions,

$$\xi \longrightarrow \infty : \frac{d\overline{T}}{d\xi} \longrightarrow 0 \, , \quad \frac{d\bar{u}}{d\xi} \longrightarrow 0. \tag{18}$$

where in deriving (17) we have assumed that

$$\bar{\rho} = \frac{\rho}{\rho_b} = \frac{T}{T_b} = \overline{H}$$

In the special case where we have considered $\widetilde{P}_r = 1$, it is convenient to define,

$$\overline{L} = \frac{C_p \, T + \tfrac{1}{2} \, u^2}{u_b^2} = \frac{\overline{T}}{(\gamma - 1) \, M_b^2} + \frac{\overline{T}}{2 \, M^2 H_b} + \tfrac{1}{2} \, \bar{u}^2 \tag{19}$$

so that

$$\frac{d\overline{L}}{d\xi} = \frac{\dfrac{k \, dT}{dx}}{\rho_b \, u_b^3} + \frac{\dfrac{u \, \widetilde{\mu} \, du}{dx}}{\rho_b \, u_b^3} = \frac{- q + u \, \tau}{\rho_b \, u_b^2} \tag{20}$$

Further defining,

$$Q = \overline{L} - \overline{L}_b \tag{21}$$

the equation (17) can be written as,

$$\frac{dQ}{d\xi} - Q = \left(\frac{dQ}{d\xi}\right)_b \tag{22}$$

where by definition, $Q = 0$, at $\xi = 0$ and from the boundary conditions (18), $\frac{dQ}{d\xi} \to 0$ as $\xi \to \infty$. The only possible solution is, obviously,

$$Q \equiv 0 \equiv \frac{dQ}{d\xi} = \frac{d\bar{L}}{d\xi} \tag{23}$$

Hence (20) to (22) give

$$1 + \frac{\gamma - 1}{2} M_b^2 \left[1 - \bar{u}^2 + \frac{1}{M^2 H_b}\right] = \bar{T}\left[1 + \frac{\gamma - 1}{2}\frac{M_b^2}{M^2 H_b}\right] \tag{24}$$

and

$$q = u\tau \tag{25}$$

Equation (16) then becomes,

$$\frac{d\bar{u}}{d\xi} = P\bar{u} - Q + \frac{R}{\bar{u}} + \frac{S}{\bar{u}2}. \tag{26}$$

where by definition,

$$\bar{u} = 1 \quad \text{as} \quad \xi \to 0 \tag{27}$$

and

$$\frac{d\bar{u}}{d\xi} \to 0 \text{ as } \xi \to \infty. \tag{28}$$

$$P = \frac{\frac{\gamma + 1}{2\gamma} + \frac{\gamma - 1}{2}\frac{M_b^2}{M^2 H_b}}{1 + \frac{\gamma - 1}{2}\frac{M_b^2}{M^2 H_b}} \tag{29}$$

$$Q = 1 + \frac{1}{2} C_{h_c} + \frac{1}{\gamma M_b^2} + \frac{1}{2 M^2 H_b} \tag{30}$$

$$R = \frac{\frac{1}{\gamma M_b^2} + \frac{\gamma - 1}{2\gamma}\left(1 + \frac{1}{M^2 H_b}\right)}{1 + \frac{\gamma - 1}{2}\frac{M_b^2}{M^2 H_b}} \tag{31}$$

and

$$S = \frac{1}{2 M^2 H_b} \tag{32}$$

For integrating (26) we put it in the following form

$$\int \frac{\bar{u}^2 \, d\bar{u}}{(\bar{u} - e_1)(\bar{u} - e_2)(\bar{u} - e_3)} = \int P \, d\xi + \text{Constant.} \tag{33}$$

if the roots of

$$\bar{u}^3 - \frac{Q}{P}\bar{u}^2 + \frac{R}{P}\bar{u} + \frac{S}{P} = 0 \tag{34}$$

are all real. In case one root of (34) is real and other two are complex, (26) can be written as

$$\int \frac{\bar{u}^2 \, d\bar{u}}{(\bar{u}^2 + \alpha\bar{u} + \beta)(\bar{u} - \gamma)} = \int P \, d\xi + \text{Constant,} \tag{35}$$

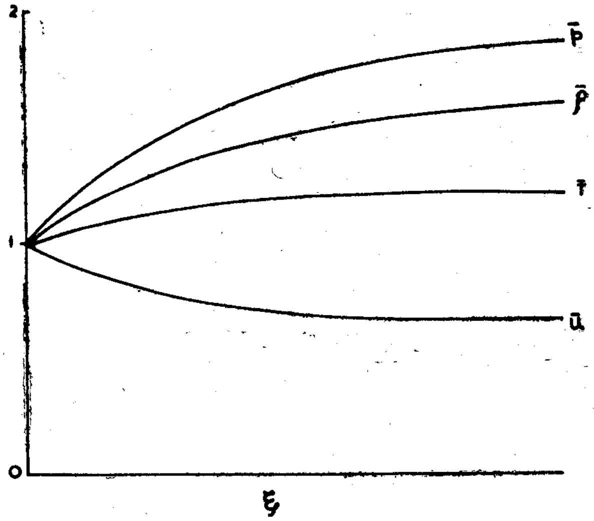

Fig. 1—Variation of pressure density, temperature and velocity within a boundary shock for $\gamma = 5/3$, $M_b = 1$, $C_{hc} = \frac{1}{2}$; $MH_b = 10$; $\bar{P}_r = 1$.

TABLE 1

VALUES OF $\bar{\rho}$, $\bar{T}$, $\bar{p}$ CORRESPONDING TO REPRESENTATIVE VALUES OF $\bar{u}$

$\bar{u}$	$\bar{\rho}$	$\bar{T}$	$\bar{p}$
·6	1·6	1·21	1·93
·7	1·4	1·16	1·62
·8	1·2	1·11	1·33
·9	1·1	1·06	1·16
1·0	1·0	1·0	1·0

The form (35) is however not admissible in the physical problem we are encountered with because the value of $\bar{u}$ cannot be complex, it being the ratio between the velocities in front and behind the boundary shock. Then, integrating (33) in the usual manner and applying the condition (27) we have

$$\left(\frac{\bar{u}-e_1}{1-e_1}\right)^{\frac{e_1^2}{(e_1-e_2)(e_1-e_3)}} \left(\frac{\bar{u}-e_2}{1-e_2}\right)^{\frac{e_2^2}{(e_2-e_1)(e_2-e_3)}} \left(\frac{\bar{u}-e_3}{1-e_3}\right)^{\frac{e_3^2}{(e_3-e_1)(e_3-e_2)}} = e^{P\xi}. \qquad (36)$$

In order to have a qualitative picture of the (36), we take

$$\gamma = \frac{5}{3}\,,\ C_{hc} = \frac{1}{2}\,,\ M_b = 1\,,\ MH_b = 10.$$

and substitute them in the relations (29) to (32) to obtain values of P, Q, R and S. Then (34) becomes

$$\bar{u}^3 - 2{\cdot}3\,\bar{u}^2 + \bar{u} + {\cdot}006 = 0 \qquad (37)$$

Solving the equation numerically, its roots come out to be approximately ·6, 1·7 and —·006. The equation (36) then gives

$$\left(\frac{\bar{u}-{\cdot}6}{{\cdot}4}\right)^{-{\cdot}54} \left(\frac{\bar{u}+{\cdot}006}{1{\cdot}006}\right)^{{\cdot}000036} \left(\frac{\bar{u}-1{\cdot}7}{-{\cdot}7}\right)^{1{\cdot}5} = e^{{\cdot}8\xi} \qquad (38)$$

Evidently when $\bar{u} = 1$; $\xi \to 0$ and when $\bar{u} = {\cdot}6$, $\xi \to \infty$. For values of $\bar{u}$ between 1 and ·6, ξ takes steadily large values. There is no need of finding the values of ξ corresponding to $\bar{u}$ greater than 1 and negative values of $\bar{u}$ since they are inconsistant with the physical problem under consideration. Keeping in mind that

$$\bar{\rho} = \frac{\rho}{\rho_b} = \frac{1}{\bar{u}} \qquad (39)$$

$$\bar{p} = \frac{p}{p_b} = \bar{\rho}\,\bar{T} \qquad (40)$$

the corresponding values of $\bar{\rho}$, $\bar{T}$ and $\bar{p}$ are given in the Table 1 and the variations of these quantities with ξ are plotted in the Fig. 1.

REFERENCES

1. MARTIN, E. D., *J. Fluid Mech.*, 28 (1967), 337.

2. VERMA, B. G. & PRASAD, B., *Pure Appl. Geophys.*, **87** (1971), 207.