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Transient Thermal Diffusion in Conical Bodies

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Abstract. A numerical solution has been obtained for transient thermal diffusion in a cone in which chemical, electrical or nuclear energy is converted into thermal energy at a constant rate. An implicit method is used to set up the finite difference equations and detailed analysis is carried out to trace the time history of the temperature distribution from the initial stages to the steady state. The effect of the rate of heat generation on the time required to reach steady state thermal distribution has also been depicted.

Nomenclature

c	Specific heat of the material of the cone
H	Coefficient of heat conductance
K	Thermal conductivity
L	Length of the cone
P	Perimeter of the cone
Q	Heat generation per unit volume per unit time
$T_{i,m}$	Non-dimensional temperature at the ith nodal point and mth time step
t	Non-dimensional time (Fourier number)
V	Temperature within the cone
V_0	Temperature of the cone at $X = L$
V_∞	Temperature of the surrounding medium
W	Area of cross-section of the cone

Greek Symbols

α	Thermal diffusivity
ρ	Density of the material of the cone
θ	Semi-vertical angle of the cone
τ	time

1. Introduction

Transient thermal diffusion is very important in atmospheric, earth, biological and technological sciences. Structural technology relies heavily on transient thermal diffusion studies in material selection, for example in solid-fuel rocket nozzles, in reentry shields, in chemical and thermal reactor components and in combustion devices.

This investigation deals with the study of thermal diffusion in a conical body which models a reactor. The reactor wall is not insulated and its diameter is supposed to be tapering slowly to a point and consequently the temperature can be considered to be essentially independent of the radius of the cone. The problem has been solved by assuming an effective thermal conductivity of the packing. The aim is to find out as to how the temperature distribution in the configuration evolves with time and subsequently reaches a steady state when heat is produced within the body at a constant rate. It has also been pointed out as to how the rate of heat generation within the body affects the thermal distribution and retards the acquisition of steady state.

The study of thermal diffusion in cylindrical bodies has been carried out by Carslaw and Jaeger[1] and Meyer[2]. However, the analytic methods involve grappling with Bessel Functions, because of the cylindrical shapes. The analytic study will become all the more complicated in the situations where heat is generated within the body which is not a very uncommon phenomena as evinced in the studies of chemical and nuclear reactors.

The problem of thermal diffusion in a slender cone with specified rate of heat generation has been solved in the present paper using Crank-Nicholson's[3] implicit finite difference scheme. The transient thermal diffusion equation along with the initial and boundary conditions transform to a tri-diagonal system of linear equations which have been solved to trace the time history of the temperature distribution within the cone.

2. Formulation of the Problem

Let the slender cone extends from $X = 0$ to L. The cone wall is not insulated and is thin enough to allow a one-dimensional analysis. Let the effective thermal conductivity of the packing and the conical shell be K and the coefficient of heat transfer between the lateral surface of the body and its surrounding be H.

The differential equation governing the temperature distribution, say V, in the cone is obtained by applying the principle of conservation of energy viz., the sum total of the rate of energy increase across the surface of the cone and the rate of generation of energy within it is equal to the rate of change of energy stored in the mass of the volume. This gives the following parabolic equation

$$\frac{\partial}{\partial X}\left(WK \frac{\partial V}{\partial X}\right) - H(V - V_{\infty})\, P \sec\theta + QW = \rho c W \frac{\partial V}{\partial \tau} \quad (1)$$

where the first term results from the transfer of heat energy along the axis of the cone, the second term is a consequence of the heat lost from the lateral surface to the environment which is supposed to be at temperature V_∞, the third term is introduced because of the heat generation which is supposed to be Q per unit volume per unit time and the right hand side is the rate of change of the heat content of the cone. W is the area of cross-section of the cone and is given by

$$W = \pi(X \tan \theta)^2$$

and P is the perimeter of the section and is given by $2\pi X \tan \theta$, where θ is the semi-vertical angle of the cone.

The Eqn. (1) governing the transient behaviour of the temperature distribution can be simplified to give

$$\frac{\partial^2 V}{\partial X^2} + \frac{2}{X}\frac{\partial V}{\partial X} + \frac{Q}{K} - \frac{2H(V - V_\infty)\operatorname{cosec}\theta}{KX} = \frac{1}{\alpha}\frac{\partial V}{\partial \tau} \tag{2}$$

where $\alpha \left(= \frac{K}{\rho c}\right)$ is the thermal diffusivity of the cone.

The boundary condition at $X = 0$ is taken as

$$\frac{\partial V}{\partial X} = \frac{H \operatorname{cosec}\theta}{K}(V - V_\infty) \tag{3}$$

to ensure that the contribution from the second term of Eqn. (2) remains finite at $X = 0$. This, however, does not result in any loss of generality because Eqn. (3) is Newton's law of convection with slightly modified constant of proportionality.

The circular face of the cone is maintained at a constant temperature, i.e.

$$V(L, \tau) = V_0 \text{ for all } \tau \tag{4}$$

The conical rod is supposed to be initially at the temperature V_∞, i.e.,

$$V(X, 0) = V_\infty \text{ for all } X \tag{5}$$

Using the transformations

$$x = \frac{X}{L},\ T = \frac{V - V_\infty}{V_0 - V_\infty} \text{ and } t = \frac{\alpha\tau}{L^2}$$

the set of Eqns. (2) – (5) can be transformed to the following non-dimensional form

$$\frac{\partial^2 T}{\partial x^2} + \frac{2}{x}\left(\frac{\partial T}{\partial x} - AT\right) = \frac{\partial T}{\partial t} - p \tag{6}$$

where $A = \dfrac{HL \operatorname{cosec}\theta}{K}$ and $p = \dfrac{QL^2}{K(V_0 - V_\infty)}$

$$\frac{\partial T}{\partial x} - AT = 0 \text{ for all } t \text{ and } x = 0 \tag{7}$$

$$T(1, t) = 1 \text{ for all } t \tag{8}$$

$$T(x, 0) = 0 \text{ for all } x \text{ and } t = 0 \tag{9}$$

The finite difference analogue of the system of Eqns. (6) – (9), their solution and discussion of the numerical results are contained in the subsequent section.

3. Solution of the Problem

The finite difference equations are set up by spacing the first point one-half an increment away from the boundary. The value of the independent variable at different nodal points is given by

$$x_i = (i - \tfrac{1}{2})\,\Delta x$$

where $\Delta x = \dfrac{1}{n}$, n being the number of increments.

Using Crank-Nicholson's implicit scheme and denoting the non-dimensional temperature at ith nodal point and mth time step by $T_{i,m}$, the finite difference analogue of Eqn. (6) is

$$\left(\frac{2i-3}{2i-1}\right) T_{i-1,m+1} - \left[2 + \frac{4A\Delta x}{2i-1} + \frac{2(\Delta x)^2}{\Delta t}\right] T_{i,m+1} + \left(\frac{2i+1}{2i-1}\right) T_{i+1,m+1} = -\left(\frac{2i-3}{2i-1}\right) T_{i-1,m} - \left(\frac{2i+1}{2i-1}\right) T_{i+1,m} + \left[2 + \frac{4A(\Delta x)}{2i-1} - \frac{2(\Delta x)^2}{\Delta t}\right] T_{i,m} \tag{10}$$

for $2 \leqslant i \leqslant n-1$.

This gives a set of $n-2$ equations for n nodal points. However, the set of equations can be made complete by using the Eqns. (7) and (8), which yield the following two finite difference equations respectively

$$\left[2 + 4A(\Delta x) + \frac{2 - A(\Delta x)}{2 + A(\Delta x)} + \frac{2(\Delta x)^2}{\Delta t}\right] T_{1,m+1} - 3T_{2,m+1} = 3T_{2,m} - \left[2 + 4A(\Delta x) + \frac{2 - A(\Delta x)}{2 + A(\Delta x)} - \frac{2(\Delta x)^2}{\Delta t}\right] T_{1,m} \tag{11}$$

$$\left(\frac{2n-3}{2n-1}\right) T_{n-1,m+1} - \left[2 + \frac{4A(\Delta x)}{2n-1} + \frac{2n+1}{2n-1} + \frac{2(\Delta x)^2}{\Delta t}\right] T_{n,m+1} = -\left(\frac{2n-3}{2n-1}\right) T_{n-1,m} + \left[2 + \frac{4A(\Delta x)}{2n-1} + \frac{2n+1}{2n-1} - 2\frac{(\Delta x)^2}{\Delta t}\right] T_{n,m} - 4\left(\frac{2n+1}{2n-1}\right) - p \tag{12}$$

It may be observed the right hand side of the Eqns. (10), (11) and (12) are known at $t = 0$ through Eqns. (8) and (9). Thus they form a tri-diagonal system of n linear algebraic equations which can be solved by various available techniques. However, we use Thomas' algorithm for the solution of these equations which is very well suited for tri-diagonal system of linear equations.

A computer algorithm has been developed which can cater for different step sizes of x. The size of time interval is increased as the solution progresses with time. As the finite difference equations are set up with the help of Crank-Nicholson implicit method, which is second order correct, the value of n is taken as 20 which gives result correct upto third decimal place.

Numerical results and response of the thermal distribution to the various physical parameters are discussed in the next section.

4. Results and Discussions

The effect of the rate of heat generation on the acquisition of the steady state temperature distribution is shown in Figs. 1 and 2.

Fig. 1 shows as to how the steady state temperature distribution in the cone varies from the tip of cone towards the plane face. It is observed that the peak temperatures are observed predominantly in the right half of the cone. Also the temperature in the cone rises with the rise in the rate of heat generation which is as expected and this adds to the confidence in the numerical procedure. The location of the peak temperature drifts away from the plane surface with the increase in rate of heat generation.

Fig. 2 shows the variation in the time taken for the thermal distribution to evolve to a steady state when the rate of heat generation is varied. It is observed that higher rate of heat generation retards the acquisition of steady state.

Figs. 3 and 4 depict the evolution of thermal distribution with time.

Fig. 3 shows the distribution of temperature in the cone at different times when there are no heat sources. It is observed that as the time elapses, the temperature of the cone increases and settles down to a steady state at $t = 0.2$ as evinced by Fig. 2.

Fig. 4 shows the distribution of temperature in the cone at different times when there is heat generation within the cone.

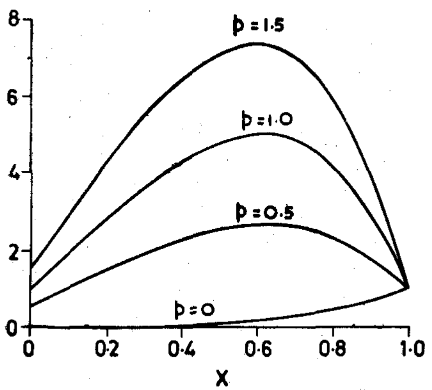

Figure 1. Steady state temperature for different rates of heat generation.

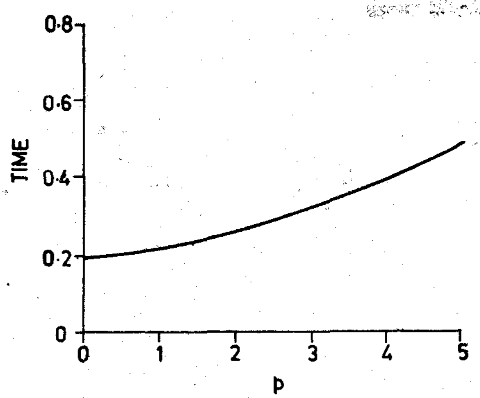

Figure 2. Time required to reach steady state vs. rate of heat generation.

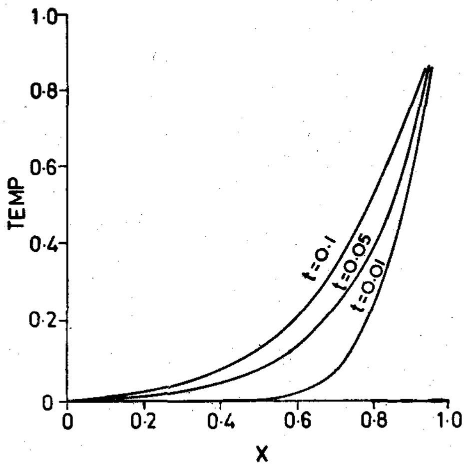

Figure 3. Time history of temperature without heat sources.

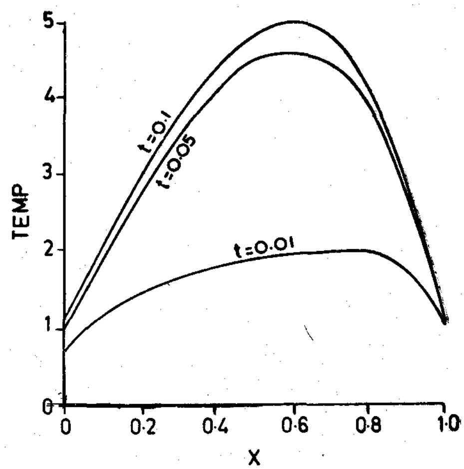

Figure 4. Time history of temperature with heat sources $p = 1.0$.

These few samples curves show how the temperature distribution in a cone evolves from initial stages to the steady state. The effect of the rate of heat generation on the thermal behaviour has also been clearly brought out. However, the method can be used to get the results for the complete spectrum of the values of the various parameters involved.

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