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Bending of Composite Plates—I

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Abstract. The problems of bending of aelotropic incompressible composite circular blocks into ellipsoidal shell have been studied following the method of Seth, *et al.* and a particular case has been obtained.

1. Introduction

The theory of finite strain has been developed on the hypothesis that the second order terms may not be neglected in the components of strain. Very few attempts have been made to obtain solutions for the finite bending of composite plates into shells. The problems of bending of plates have been considered by Seth[1,2], Lakshminarayana & Kesava Rao[3], when the material is isotropic and aelotropic. Following the Seth's method, Lakshminarayana[4] considered the problem of bending of composite circular plate into spherical shell when the material is isotropic. Rama Rao, S., et al[5] considered the problem of bending of incompressible circular block into ellipsoidal shell when the material is aelotropic following the method of Green & Adkins[6]. In this paper this method is followed to solve the problem of bending of aelotropic incompressible composite circular block into an ellipsoidal shell. The problem of bending of a incompressible composite circular block into spherical shell has been obtained as a particular case. The solution has been obtained in terms of a general strain energy function.

2. Notation

We adopt the notation and formulae of Green & Zerna[7], and Green & Adkins[8].

3. Bending of an Incompressible Aelotropic Composite Circular Block into an Ellipsoidal Shell

Suppose that in the undeformed state of the body it is a circular block bounded by the planes $x_3 = a_1$, $x_3 = a_2$, $x_3 = a_3$ $(a_1 < a_2 < a_3)$ and the cylinder $x_1^2 + x_2^2 = a^2$, where $x_3 = a_2$ is the common plane of the two blocks. The block is then bent symmetrically about the x_3-axis into a part of an ellipsoidal shell, whose inner, common and outer boundaries are given by the ellipsoides of revolution obtained by revolving the confocal ellipses

$$x_3 = c \cosh \xi \cos \eta,\ x_1 = c \sinh \xi \sin \eta,\ \xi = \xi_j\ (j = 1, 2, 3) \tag{1}$$

about the x_3-axis respectively and the edge $\eta = \alpha$. Let the y_i-axes coincide with the x_i-axes, and the curvilinear coordinates θ^i in the deformed state be a system of orthogonal coordinates (ξ, η, ϕ). Then we have

$$y_1 = c \sinh \xi \sin \eta \cos \phi,\ y_2 = c \sinh \xi \sin \eta \sin \phi,\ y_3 = c \cosh \xi \cos \eta \tag{2}$$

Since the deformation is symmetrical about x_3-axis, this implies[5] that

$$x_1 = F_k(\eta) \cos \phi,\ x_2 = F_k(\eta) \sin \phi,\ x_3 = f_k(\xi),\ k = 1, 2 \tag{3}$$

where $k = 1$ refers to the region bounded by the curved surfaces ξ_1 and ξ_2. Similarly $k = 2$ refers to the region bounded by the surfaces ξ_2 and ξ_3.

The metric tensors for the strained and unstrained state of the body are given by

$$(G_{ij})_k = \begin{bmatrix} c^2(\cosh^2 \xi - \cos^2 \eta) & 0 & 0 \\ 0 & c^2(\cosh^2 \xi - \cos^2 \eta) & 0 \\ 0 & 0 & c^2 \sinh^2 \xi \sin^2 \eta \end{bmatrix} \tag{4}$$

$$(g_{ij})_k = \begin{bmatrix} f_k'^2 & 0 & 0 \\ 0 & F_k'^2 & 0 \\ 0 & 0 & F_k^2 \end{bmatrix} \tag{5}$$

Where $\quad f_k' = \dfrac{df_k}{d\xi}$ and $F_k' = \dfrac{dF_k}{d\xi}$

The condition of incompressibility $I_3 = 1$ gives

$$\frac{c^2(\cosh^2 \xi - \cos^2 \eta) \sinh \xi}{f_k'} = \frac{F_k F_k'}{\sin \eta} \tag{6}$$

With the assumption that η is small[5], we get

$$\frac{c^3 \sinh^3 \xi}{f_k'} = \frac{F_k F_k'}{\eta} = A_k \tag{7}$$

where A_k are constants. From this, we get

$$X_3 = f_k(\xi) = \frac{c^3}{A_k}\left(\frac{\cosh^3 \xi}{3} - \cosh \xi\right) + B_k \tag{8}$$

and

$$x_1^2 + x_2^2 = F_k^2(\eta) = A_k\eta^2 + D_k \tag{9}$$

where B_k, D_k are constants.

As the internal, common and external boundaries of the ellipsoidal shell are given by $\xi = \xi_j$, $(j = 1, 2, 3)$ respectively which were initially the planes $x_3 = a_1$, $x_3 = a_2$, $x_3 = a_3$, The Eqn. (8) gives

$$\left.\begin{aligned} a_1 &= \frac{c^3}{A_1}\left(\frac{\cosh^3 \xi_1 - 3\cosh \xi_1}{3}\right) + B_1 \\ a_2 &= \frac{c^3}{A_1}\left(\frac{\cosh^3 \xi_2 - 3\cosh \xi_2}{3}\right) + B_1 \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} a_2 &= \frac{c^3}{A_2}\left(\frac{\cosh^3 \xi_2 - 3\cosh \xi_2}{3}\right) + B_2 \\ a_3 &= \frac{c^3}{A_2}\left(\frac{\cosh^3 \xi_3 - 3\cosh \xi_3}{3}\right) + B_2 \end{aligned}\right\} \tag{11}$$

which when solved give the values of A_k and B_k.

Since the bending is symmetrical about the x_3-axis, we must have $x_1^2 + x_2^2 = 0$ when $\eta = 0$. Then Eqn. (9) gives

$$x_1^2 + x_2^2 = F_k^2(\eta) = A_k\eta^2 \tag{12}$$

The strain components are given[5] by

$$(e_{11})_k = (e_{22})_k = \tfrac{1}{2}\left(\frac{c^2 \sinh^2 \xi}{A_k} - 1\right)$$

$$(e_{33})_k = \tfrac{1}{2}\left(\frac{A_k^2}{c^4 \sinh^4 \xi} - 1\right)$$

$$(e_{ij})_k = 0 \text{ for } i \neq j \tag{13}$$

The components of stress tensor are given by

$$(T^{11})_k = \frac{A_k^2}{c^6 \sinh^6 \xi}\left(\frac{\partial W}{\partial e_{33}}\right)_k + \frac{p_k}{c^2 \sinh^2 \xi}$$

$$(T^{22})_k = \frac{1}{A_k}\left(\frac{\partial W}{\partial e_{11}}\right)_k + \frac{p_k}{c^2 \sinh^2 \xi}$$

$$(T^{33})_k = \frac{1}{A_k \eta^2} \left(\frac{\partial W}{\partial e_{11}}\right)_k + \frac{p_k}{\eta^2 c^2 \sinh^2 \xi} \tag{14}$$

$$(T^{ij})_k = 0 \text{ for } i \neq j$$

Substituting Eqn. (14) in the equations of equillibrium and solving, we get

$$p_k = W_k + (W_0)_k - \frac{A_k^2}{c^4 \sinh^4 \xi} \left(\frac{\partial W}{\partial e_{33}}\right)_k \tag{15}$$

From Eqns. (14) and (15), the physical components of stress are given by

$$(\sigma_{11})_k = W_k + (W_0)_k$$

$$(\sigma_{22})_k = (\sigma_{33})_k = W_k + (W_0)_k + \frac{c^2 \sinh^2 \xi}{A_k} \left(\frac{\partial W}{\partial e_{11}}\right)_k - \frac{A_k^2}{c^4 \sinh^4 \xi} \left(\frac{\partial W}{\partial e_{33}}\right)_k \tag{16}$$

Boundary Conditions

The stresses should be continuous across the common surface $\xi = \xi_2$ i.e.

$$(T^{11})_{k=1} = (T^{11})_{k=2}$$

If the inner and outer boundaries of the shell $\xi = \xi_1$ and $\xi = \xi_3$ are free from tractions, we must have

$$(\sigma_{11})_{k=1} = 0 \text{ when } \xi = \xi_1$$

$$(\sigma_{11})_{k=2} = 0 \text{ when } \xi = \xi_3$$

which on substitution in Eqn. (16) gives

$$(W_0)_{k=1} = -W_{k=1}(\xi_1)$$

$$(W_0)_{k=3} = -W_{k=2}(\xi_3) \tag{17}$$

On the edge $\eta = \alpha$ the distribution of tractions between ϕ and $\phi + d\phi$ give rise to a force F and a couple of moment M about the origin given by

$$F = \alpha \left[\int_{\xi_1}^{\xi_2} (\sigma_{22})_{k=1} (c^2 \sinh^2 \xi)\, d\xi + \int_{\xi_2}^{\xi_3} (\sigma_{22})_{k=2} (c^2 \sinh^2 \xi)\, d\xi \right]$$

$$M = \alpha \left[\int_{\xi_1}^{\xi_2} (\sigma_{22})_{k=1} (c^2 \sinh^2 \xi)(c \cosh \xi)\, d\xi + \int_{\xi_2}^{\xi_3} (\sigma_{22})_{k=2} (c^2 \sinh^2 \xi)(c \cosh \xi)\, d\xi \right] \tag{18}$$

Substituting Eqn. (16) in the above equation we can determine the values of F and M, when the strain energy functions $W_{k=1}$ and $W_{k=2}$ are specified for the two materials.

4. Particular Case

Bending of an Aelotropic Incompressible Circular Block into a Spherical Shell

If $c \cosh \xi_i = c \sinh \xi_i$ in Eqn. (1), we get the case of a circular block bent into a spherical shell, so that $\xi \rightarrow \infty$, $c \rightarrow 0$, and $c \cosh \xi$, $c \sinh \xi \rightarrow r$ and consequently the orthogonal curvilinear coordinates (ξ, η, ϕ) are replaced by the spherical polar coordinates (r, θ, ϕ).

Then the Eqn. (16) reduce to

$$(\sigma_{11})_k = W_k + (W_0)_k$$

$$(\sigma_{22})_k = (\sigma_{33})_k = W_k + (W_0)_k + \frac{r^2}{A_k}\left(\frac{\partial W}{\partial e_{11}}\right)_k - \frac{A_k^2}{r^4}\left(\frac{\partial W}{\partial e_{33}}\right)_k \tag{19}$$

From Eqns. (18) and (19) we obtain the resultant force F and the couple M on the edge required to bend an aelotropic circular block into part of spherical shell.

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