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Time Variation of Orbital Parameters of an Equatorial Satellite

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Abstract. A few orbital parameters which take into account earth's finite shape as well as its axial rotation are developed for a satellite moving in the earth's equatorial plane. Expressions for line of sight velocity $\frac{dr_{es}}{dt}$, variation of elevation angle $\frac{d\zeta}{dt}$, and condition for satellite's accessibility from a ground station are obtained. For a non-rotating earth equations for variation of arc length D_c and segment area A_s are derived. The various parameters are plotted against time for different orbital eccentricities.

1. Introduction

Pure conic orbits serve as sufficiently accurate models in many orbit mechanics problem. A number of equations have been derived from basic laws of Keplerian motion[1,2]. The relations were such which did not consider the rotation and dimension of the earth. This, however, did not entail any fundamental error. In the present communication some orbital parameters will be analysed which take into account the finite size of the earth as well as its axial rotation. For simplicity, it will be assumed that the satellite is moving in the plane of the equator, i.e., the angle between the orbital plane and the equatorial plane i is zero. The path of the satellite, as seen from the pole, is depicted in Fig. 1.

2. Line of Sight Velocity and Elevation Angle

It is apparent from geometrical considerations that

$$r_{es} \sin \lambda = r_s \sin \phi - r_e \sin \theta \tag{1}$$

$$r_{es} \cos \lambda = r_s \cos \phi - r_e \cos \theta \tag{2}$$

Differentiating w.r.t. time

$$r_{es} \cos \lambda \frac{d\lambda}{dt} + \sin \lambda \frac{dr_{es}}{dt} = r_s \cos \phi \frac{d\phi}{dt} + \sin \phi \frac{dr_s}{dt} - r_e \cos \theta \frac{d\theta}{dt} \tag{3}$$

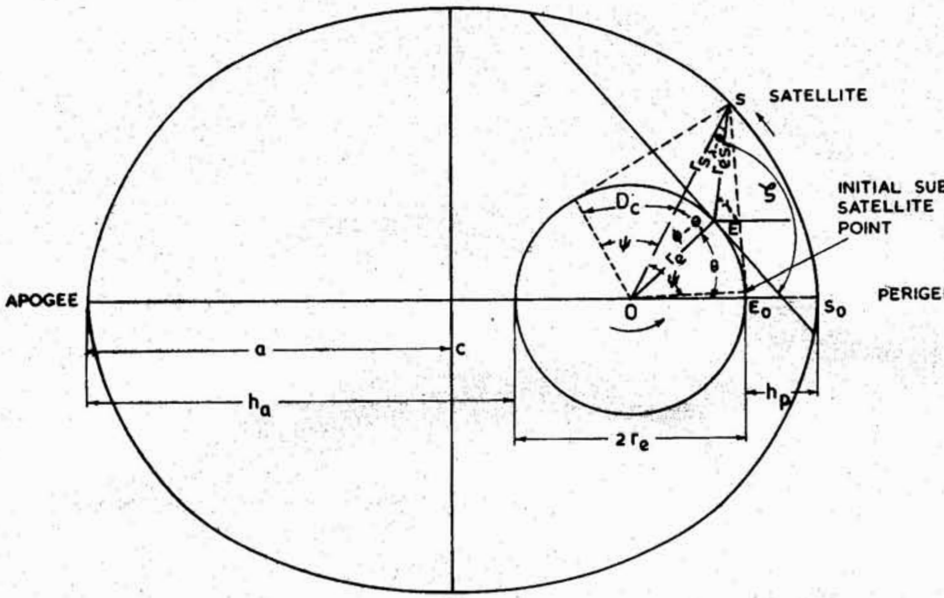

Figure 1. Top view of the elliptic orbit of a satellite in earth's equatorial plane.

$$- r_{es} \sin \lambda \frac{d\lambda}{dt} + \cos \lambda \frac{dr_{es}}{dt} = - r_s \sin \phi \frac{d\phi}{dt} + \cos \phi \frac{dr_s}{dt} + r_e \sin \theta \frac{d\theta}{dt} \tag{4}$$

Eliminating $\frac{dr_{es}}{dt}$

$$r_{es} \frac{d\lambda}{dt} = r_s \cos (\lambda - \phi) \frac{d\phi}{dt} - \sin (\lambda - \phi) \frac{dr_s}{dt} - r_e \cos (\lambda - \theta) \frac{d\theta}{dt} \tag{5}$$

Using the following relations

$$r_{es} \cos (\lambda - \phi) = r_s - r_e \cos (\phi - \theta) \tag{6}$$

$$r_{es} \cos (\lambda - \theta) = r_s \cos (\phi - \theta) - r_e \tag{7}$$

$$\frac{\sin (\lambda - \phi)}{r_e} = \frac{\sin (\phi - \theta)}{r_{es}} \tag{8}$$

$$\frac{\sin (180 - \overline{\lambda - \theta})}{r_s} = \frac{\sin (\phi - \theta)}{r_{es}} \tag{9}$$

$$\left(\frac{r_{es}}{r_e}\right)^2 = \left(\frac{r_s}{r_e}\right)^2 - 2\left(\frac{r_s}{r_e}\right) \cos (\phi - \theta) + 1 \tag{10}$$

We have on simplification

$$\frac{d\lambda}{dt} = \frac{\left[\left\{\left(\frac{r_s}{r_e}\right)^2 - \left(\frac{r_s}{r_e}\right)\cos(\phi - \theta)\right\}\frac{d\phi}{dt} + \left\{1 - \left(\frac{r_s}{r_e}\right)\cos(\phi - \theta)\right\}\frac{d\theta}{dt} - \left\{\frac{1}{r_e}\sin(\phi - \theta)\right\}\frac{dr_s}{dt}\right]}{\left[\left(\frac{r_s}{r_e}\right)^2 - 2\left(\frac{r_s}{r_e}\right)\cos(\phi - \theta) + 1\right]} \tag{11}$$

The elevation angle ζ is

$$\zeta = \lambda - \theta + \frac{\pi}{2}$$

The derivative of elevation angle is

$$\frac{d\zeta}{dt} = \frac{d\lambda}{dt} - w_{r_e} \tag{12}$$

Since $\theta = w_{r_e}\, t$, $\frac{d\theta}{dt}$ or w_{r_e} gives earth's rotational rate.

Eliminating $\frac{d\lambda}{dt}$ from Eqns. (3) and (4)

$$\frac{dr_{es}}{dt} = r_s \sin(\lambda - \phi)\frac{d\phi}{dt} + \cos(\lambda - \phi)\frac{dr_s}{dt} - r_e \sin(\lambda - \theta)\frac{d\theta}{dt} \tag{13}$$

Making use of the Eqns. (6) to (10)

$$\frac{dr_{es}}{dt} = \frac{r_e \sin(\phi - \theta)\left(\frac{d\phi}{dt} - \frac{d\theta}{dt}\right)\left(\frac{r_s}{r_e}\right) - \left\{\left(\frac{r_s}{r_e}\right) - \cos(\phi - \theta)\right\}\frac{dr_s}{dt}}{\left[\left(\frac{r_s}{r_e}\right)^2 - 2\left(\frac{r_s}{r_e}\right)\cos(\phi - \theta) + 1\right]^{1/2}} \tag{14}$$

Now r_{es} represents the 'slant range' and its differential coefficient $\frac{dr_{es}}{dt}$ gives the line of sight velocity. The elevation angle ζ and $\frac{dr_{es}}{dt}$ are important in satellite guiding and tracking. It may be noted that when elevation angle is small the effectiveness of the radar is greatly reduced due to earth's surface and atmosphere. Optical instruments too show similar limitations at low altitudes.

3. Orbital Relations

For calculation of $\frac{d\lambda}{dt}$, $\frac{dr_{es}}{dt}$ etc., we need the following relations:

$$h_a - h_p = 2ae \tag{15}$$

$$h_a + h_p + 2r_e = 2a$$

$$h_a = \frac{2r_e e}{1-e} + h_p\left(\frac{1+e}{1-e}\right) \tag{16}$$

$$r_s = \frac{a\,(1-e^2)}{1+e\cos\phi} \tag{17}$$

$$r_s^2\,\frac{d\phi}{dt} = h \tag{18}$$

$$h = \{\mu a\,(1-e^2)\}^{1/2} \tag{19}$$

From Eqns. (17), (18) and (19)

$$t = \frac{a^{3/2}\,(1-e^2)^{3/2}}{\mu}\int\frac{d\phi}{(1+e\cos\phi)^2} \tag{20}$$

$$t = \left(\frac{h_a+h_p+2r_e}{2}\right)^{3/2}\frac{\cos^{-1}\left(\dfrac{e+\cos\phi}{1+e\cos\phi}\right)}{\mu} - \left(\frac{h_a+h_p+2r_e}{2}\right)^{3/2}\frac{e\,(1-e^2)^{1/2}\sin\phi}{\mu\,(1+e\cos\phi)} \tag{21}$$

From Eqns. (15), (17), (18) and (19)

$$\frac{d\phi}{dt} = \frac{\mu\,(1+e\cos\phi)^2}{\left(\dfrac{h_a+h_p+2r_e}{2}\right)^{3/2}(1-e^2)^{3/2}} \tag{22}$$

Also, we have

$$\frac{dr_s}{dt} = \frac{eh\sin\phi}{a(1-e^2)} \tag{23}$$

or

$$\frac{dr_s}{dt} = \frac{e\sqrt{\mu}\sin\phi}{(1-e^2)^{1/2}\left(\dfrac{h_a+h_p+2r_e}{2}\right)^{1/2}} \tag{24}$$

4. Visibility of Satellite

The satellite is visible from station E, if it remains above the latter's horizon. When the satellite is rising or setting

$$r_s^2 = r_{es}^2 + r_e^2 \tag{25}$$

Using Eqn. (10), the condition for satellite's visibility is

$$\frac{r_s}{r_e}\cos(\phi-\theta) \geqslant 1 \tag{26}$$

Atmospheric refraction, however, has not been considered in arriving at this result.

5. ...ation of Arc and Segment Area

Now we will develop some parameters for a non-rotating earth. The total arc length of the earth accessible to the satellite at a particular instant is given by

$$D_c = 2\psi r_e \tag{27}$$

Since $\cos\psi = \dfrac{r_e}{r_s}$, we have

$$\cos\psi = \frac{r_e(1 + e\cos\phi)}{a(1 - e^2)} \tag{28}$$

$$\frac{d\psi}{dt} = \frac{r_e e \sin\phi}{a(1 - e^2)\left\{1 - \dfrac{1}{(r_s/r_e)^2}\right\}^{1/2}} \frac{d\phi}{dt} \tag{29}$$

$$\frac{dD_c}{dt} = \frac{2r_e^2\, e\sin\phi\left(\dfrac{r_s}{r_e}\right)\dfrac{d\phi}{dt}}{\left\{\dfrac{h_a + h_p + 2r_e}{2}\right\}(1 - e^2)\left\{\left(\dfrac{r_s}{r_e}\right)^2 - 1\right\}^{1/2}} \tag{30}$$

The area of the segment A_s is given by

$$A_s = 2\pi r_e^2(1 - \cos\psi)$$

$$\frac{dA_s}{dt} = \frac{2\pi r_e^3\, e\sin\phi}{\left(\dfrac{h_a + h_p + 2r_e}{2}\right)(1 - e^2)} \frac{d\phi}{dt} \tag{31}$$

It is apparent from Eqns. (30) and (31) that D_c and A_s have either a maxima or minima for $\phi = 0, \pi, 2\pi$ etc.

6. Results and Discussions

The numerical value of the constants used in the above equations are as follows.

$$\mu = 1.40775 \times 10^{16}\ \text{ft}^3/\text{sec}^2$$

$$w_{r_e} = 7.2921158 \times 10^{-5}\ \text{rad/sec}$$

$$r_e = 20.926428 \times 10^6\ \text{ft}$$

$$h_a = 100\ \text{nmi}$$

$$i = 0^\circ$$

The variation of $\dfrac{dr_{es}}{dt}$ or line of sight velocity vs. non-dimensionalised time t/T is depicted in Fig. 2, for $e = 0$, $e = 0.2$ and $e = 0.4$, where T is time for one complete

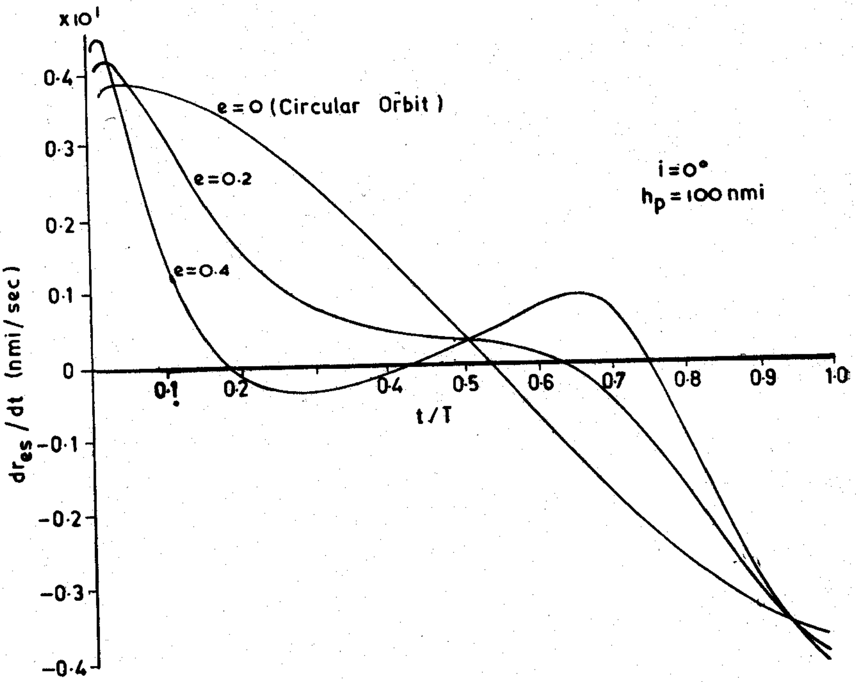

Figure 2. Line of sight velocity vs. non-dimensionalised time for different eccentricities.

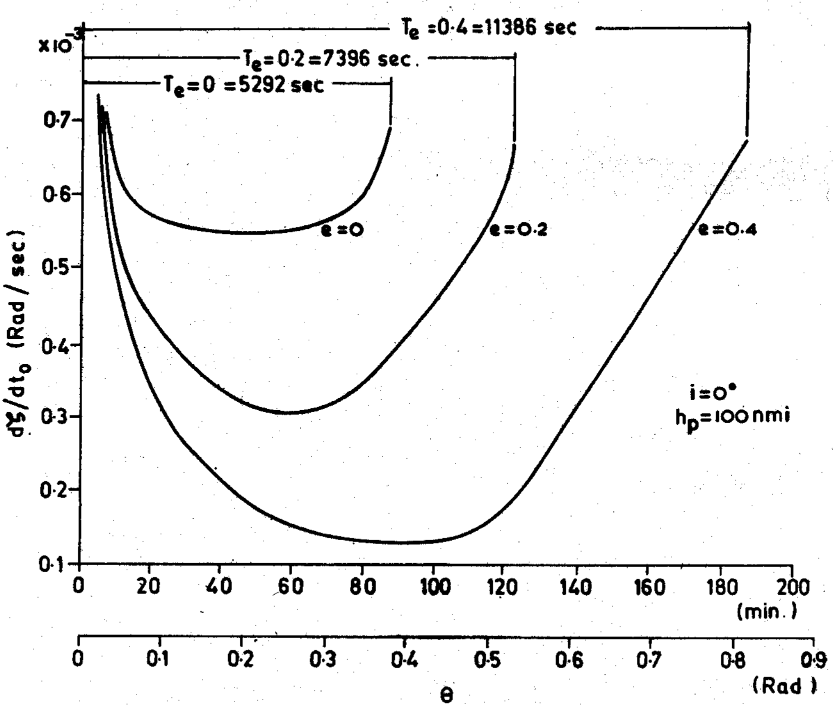

Figure 3. Rate of change of angle of elevation against time (or earth's angle of turn) for $e = 0$, 0.2 and 0.4.

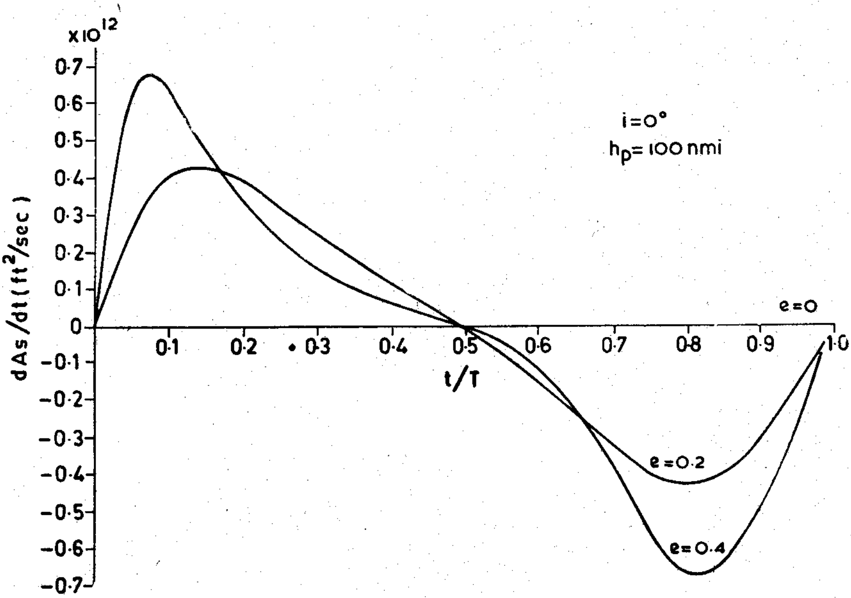

Figure 4. Rate of change of area vs. non-dimensionalised time for different eccentricities.

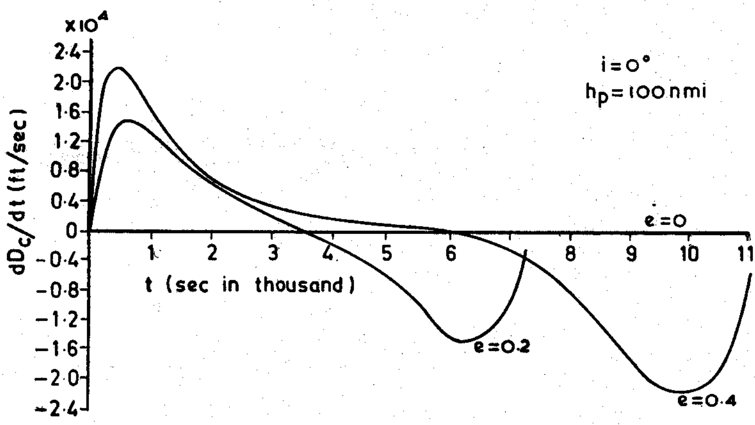

Figure 5. Rate of change of arc length against time for $e = 0$, 0.2 and 0.4.

revolution and t is time from perigee passage. The true anomaly or vectorial angle ϕ and the angle θ through which earth turns in time t are also measured from perigee. In Fig. 3 the rate of change of elevation angle $\frac{d\zeta}{dt}$ is plotted against time, t (or the angle θ through which earth turns in this period) for different orbital eccentricities. Fig. 4 provides the variation of $\frac{dA_s}{dt}$ with nondimensionalised time $\frac{t}{T}$, for different eccentricities. Similarly Fig. 5 shows the variation of $\frac{dD_c}{dt}$ with time, for $e = 0$, 0.2 and 0.4. It may be stated that Figs. 4 and 5 correspond to the case of a non-rotating earth. It is also evident from Fig. 4 that segment area A_s is maximum or minimum for $\phi = 0$,

π, 2π etc. Also from Fig. 1 it follows intutively that when $\phi = 0, 2\pi, 4\pi$ etc. A_s and D_c are minimum and they are maximum for $\phi = 0, \pi, 3\pi$ etc.

In this communication certain parameters have been discussed for satellite confined to the plane of the earth's equator. The more general case in which the orbit is inclined to the equator will be dealt with later on. In case of earth the rotation can be treated as uniform. It would be interesting to investigate the variation of various parameters when the central body rotates non-uniformly, that is it shows differential rotation like the Sun, Jupiter etc.

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References

1. Jensen, J., Townsend, G. E., Kraft, J. P. & Kork, J., 'Design Guide to Orbital Flight' (McGraw Hill, New York), 1962, Chapter 2, pp. 9–81.
2. Wolverton, R. W., 'Flight Performance Handbook for Orbital Operations' (John Wiley, New York), 1963, Chapter 2, pp. 1–99.