Def Sci J, Vol 31, No 3, July 1981, pp 251-254

Flow of an Oldroyd Fluid Past a Sphere at Low Reynolds Number

P. D. VERMA & R. C. CHAUDHARY

Department of Mathematics, University of Rajasthan, Jaipur-302004

Received 28 March 1980

Abstract. Slow motion of an Oldroyd fluid past a sphere has been investigated using matched asymptotic technique. The drag experienced by the sphere has been obtained from the solution. To the first order in τ (a non-Newtonian parameter), the drag is unchanged from its value for a Newtonian fluid but to the second order the drag is decreased.

1. Introduction

The solution for a sphere moving slowly in viscous fluid was first given by Stokes[1]. The solution gives good results in the close region of the sphere but does not explain the pattern at the large distance from the sphere. Oseen[2] improved his results so that a correct overall picture of the flow field is obtained. Proudman and Pearson[3] in his paper improved this theory and obtained the improved drag formula for the sphere by the method of matched asymptotic expansion. They distinguished the Stokes flow near the obstacle and the Oseen flow away from the obstacle. Leslie and Tanner[4] have extended the Stokes flow using the Oldroyd model and obtained a solution by expanding in terms of a non-Newtonian parameters τ Caswell and Schwarz[5] have investigated the creeping motion of Rivlin-Ericksen fluid past a sphere using the matching procedure given by Proudman and Pearson[3].

The main aim of this paper is to investigate the Caswell and Schwarz's problem by taking the Oldroyd[6] type fluid, that is, to extend the problem of Leslie and Tanner by matched asymptotic expansion. The Oldroyd fluid is of the type poly-isobutylene in carbon tetrachloride.

2. Formulation of the Problem

We shall work through the spherical polar coordinates (R, θ, ϕ) with the centre of the sphere as origin and $\theta = 0$ in the upstream direction. Owing to axial symmetry, the system is independent of ϕ coordinate. First of all we transfer all the equations, that is, the equation of state relating the stress tensor and the rate of strain tensor (Oldroyd fluid) and the familiar equations of motion and continuity for steady, incompressible flow, from tensor form to the spherical polar coordinate system and then made them non-dimensionalized by the usual procedure.

The boundary conditions are obtained by the requirement of the no-slip at the surface of the sphere and the uniform condition at infinity, that is

$$u_r = u_\theta = 0 \qquad \text{at } r = 1$$

$$u_r = -\cos\theta,\ u_\theta = \sin\theta \text{ at } r \to \infty \tag{1}$$

(u_r, u_θ etc. are velocity components non-dimensionalizad by the uniform velocity U).

3. Method of Solution

The equation of continuity suggests the existence of the stream function ψ such that

$$u_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta},\ u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{2}$$

Now for small Reynold's number, solutions for all physical components (namely stream function, stress etc.) are assumed to have an inner (Stokes) expansion as

$$\psi = \psi_0(r, \theta) + Re\,\psi_1(r, \theta) + \ldots \tag{3}$$

This is the solution of Stokes region where $Re\, r < 0(1)$. Since the Eqn. (3) does not hold good for large values of r, the uniform stream condition at infinity must be replaced by the requirement that the expansion should be perfectly matched to an expansion which is valid in the outer region. Now for the outer (Oseen) region, we define the stretched variables ρ and Ψ as

$$\rho = Re\ r \text{ and } \Psi = Re^2\psi \tag{4}$$

The expansion in the outer region, which we call the Oseen expansion, is now assumed to take the form

$$\Psi = \Psi_0(\rho, \theta) + Re\,\Psi_1(\rho, \theta) + \ldots \tag{5}$$

The solution (3) will satisfy the no-slip condition at the surface of the sphere and Eqn. (5) will satisfy the uniform condition at infinity. In order to find the complete solution the remaining constants are evaluated by matching conditions in the overlapping region.

Leading terms of the solutions

Using equation of state, motion and continuity and with the help of Eqns. (2) and (3), the equation governing ψ_0 is obtained and solved with the help of power series in terms of τ (a non-dimensional non-Newtonian parameter) as

$$\psi_0 = \psi_{00} + \tau\psi_{01} + \tau^2\,\psi_{02} + \ldots \tag{6}$$

Adopting the same procedure as given by Proudman and Pearson the solutions of various components in Eqn. (6) are given as

$$\left.\begin{aligned}\psi_{00} &= \frac{1}{4}\left(2r^2 - 3 + \frac{1}{r}\right)(1-\mu^2)\\ \psi_{01} &= \frac{3}{8}(1-\epsilon)\left(1-\frac{1}{r}\right)^3 \mu(1-\mu^2)\end{aligned}\right\} \tag{7}$$

where $\mu = \cos\theta$ and ϵ is a dimensionless constant. We have not mentioned here the solution of ψ_{02} due to its lengthy form. Hence we can have ψ_0.

The leading term for the Oseen's region can also be derived as pointed out by Proudman and Pearson and can be given as

$$\Psi_0 = \tfrac{1}{2}\ \rho^2(1-\mu^2) \tag{8}$$

Higher order terms in the Oseen and Stokes expansions

Since the matching procedure for the inner and outer expansions involves only the Newtonian terms, the first order solution to Oseen's expansion can be obtained as

$$\Psi_1 = -\tfrac{3}{2}(1+\mu)\ \left[1 - \exp\left\{-\tfrac{1}{2}\ \rho(1-\mu)\right\}\right] \tag{9}$$

The solution for ψ_1 (refer Caswell and Schwarz) which is properly matched with Ψ_1, is given as

$$\psi_{10} = \frac{3}{32}\left(2r^2 - 3r + \frac{1}{r}\right)(1-\mu^2) - \frac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)\mu(1-\mu^2) \tag{10}$$

where ψ_1 has been taken in the form

$$\psi_1 = \psi_{10} + \tau\psi_{11} + \ldots \tag{11}$$

4. Drag on the Sphere

The drag on the sphere is given by

$$\frac{D}{2\pi\eta_0 U a} = \int_0^{\pi} (p_{r\theta})_{r=1} \sin^2\theta\, d\theta + \int_0^{\pi} (p - p_{rr})_{r=1} \sin\theta\cos\theta\, d\theta \tag{12}$$

where η_0 is constant having the dimensions of viscosity, U is uniform fluid velocity, a is radius of the sphere, p is isotropic pressure and p_{rr}, $p_{r\theta}$ etc. are stress components.

Substituting the values for p, p_{rr} and $p_{r\theta}$ the final expression for the drag in our case is

$$D = 6\pi\eta_0\, Ua\left(1 + \tfrac{3}{8}\, Re\right) - [0.016(1-\epsilon)\ (3-\epsilon) - 0.618\,\beta]\,\tau^2 \tag{13}$$

where β is a known constant.

Equation (13) predicts that to the first order in τ the drag is unchanged from its value as obtained by Proudman and Pearson for a Newtonian fluid but to the second order the drag is decreased, because $\beta < 0$ and $\epsilon < 1$. If we take $\tau = 0$, the Eqn. (13) agrees with Proudman and Pearson[3].

References

1. Stokes, G. G., *Trans. Camb. Phil. Soc.*, **9** (1851) Pt. II, 8 or Coll. Papers III, 55.
2. Oseen, C. W., *Arkiv für Mathematik, Astronomi Och Fysik* No. 29, **6** (1910).
3. Proudman, I. & Pearson, J. R. A., *J Fluid Mech.* **2** (1957), 237.
4. Leslie, F. M. & Tanner, R. I., *Quart. Journ. Mech. Appld. Math.* **14** (1961), 36.
5. Caswell, B. & Schwarz, W. H., *J. Fluid Mech.*, **13** (1962), 417.
6. Oldroyd, J. G., *Proc. Roy. Soc.*, A. **245** (1958), 278.