Def Sci J, Vol 31, No. 3, July 1981, pp 231-240

Couette Flow of a Dusty Incompressible Fluid with One of the Horizontal Moving Boundaries Suddenly Stopped

V. V. RAMANA PRASAD* & N. CH. PATTABHI RAMACHARYULU

Department of Mathematics, Regional Engineering College
Warangal-506004 (A.P)

Received 25 February 1980

Abstract. The unsteady flow of an incompressible viscous fluid with a uniform distribution of dust particles between two parallel plates when one of which is impulsively stopped from the state of uniform motion is studied. Analytical expressions for velocities of the fluid and dust particles have been obtained. It is found that the dust particles slip on the wall which is brought to rest impulsively instead of sticking to it; on the other hand, they stick to the stationary plate. The slip-velocity of the dust is noticed to be decreasing as the physical time increases and the relaxation time of the dust particles decrease.

Key words. Dusty fluid; impulsive motion; flux; skin-friction; slip-velocity; relaxation time.

1. Introduction

The mechanical behaviour of dusty fluids has been a subject of study receiving greater attention during the recent past of several researchers in the field of Fluid Dynamics. Problems dealing with the influence of dust particles on viscous flows find place in several branches of Science and Technology. Some such flows are those of dissolved micromolecules, of fiber suspensions, of latex particles in emulsion-paints, of reinforcing particles in polymermelts, of rock crystals in molten lava, red corpuscles and other bodies in blood and so on.

The momentum equations given by Saffman[1] charcterizing the dusty fluid flow have been discussed by Michael and Miller[2] for the flow in the semi-infinite space over flat plate. Later several authors[3,4,5,6,8,10,11,12,13,14] examined the dusty fluid flowbetween diverse boundaries. Marble[8] and Cox & Mason[9] gave detailed reviews dealing with the growth of the subject upto that time.

In this paper, the authors study the problem of the viscous incompressible dusty fluid flow between two parallel plates one of which is kept stationary and the other is impulsively brought to rest from a state of uniform motion parallel to itself. This problem finds its application in the squeeze-film lubrication. The variation of flow

*Present address : Department of Mathematics, Gurunanak Dev Engineering College, Bidar-585403

rate of the fluid and dust particles and the skin-friction have been discussed for a wide spectrum of values of relaxation time and concentration parameters τ and n respectively. The relaxation time parameter is a representative of time scale on which velocity of dust particles adjusts itself to changes in the velocities of neighbouring fluid.

It is found thal the velocities of fluid and dust particles decrease with relaxation time τ and time t. Dust particles slip on the wall with impulsive motion instead of sticking to it. The slip-velocity decays as time increases. It is noticed that maximum velocities of fluid and dust decrease with the increase of τ and t.

2. Formulation of the Problem

Let us consider the motion of the dusty incompressible fluid between two parallel plates, $Y = a$ and $Y = -a$, which is kept stationary. The fluid is set in motion by moving the plate $Y = a$ parallel to itself with constant velocity. When the steady state is reached, the moving plate is impulsively brought to rest. The aim of the present paper is to investigate the subsequent motion. Assuming that the dust particles are uniformly distributed with a constant concentration, the equations of motion given by Saffman[1] reduce to

$$\rho \frac{\partial W}{\partial T} = \mu \frac{\partial^2 W}{\partial Y^2} + KN(W_d - W) \tag{1}$$

$$\frac{\partial W_d}{\partial T} = \frac{K}{M}(W - W_d) \tag{2}$$

where ρ is the fluid density, W is the fluid velocity between two parallel plates, W_d is the dust velocity, K is the Stokes constant, N is the dust concentration coefficient, μ is the coefficient of viscosity of fluid, M is the mass of the dust particle, and T is the time.

In terms of non-dimensional quantities defined by

$$Y = ay,\ T = \frac{\rho a^2 t}{\mu},\ (W, W_d) = \frac{\mu}{\rho a}(u, v),$$

$$N = \frac{n}{a^3},\ \tau = \frac{\mu M}{k\rho a^2},\ \lambda = \frac{KNa^2}{\mu}, \tag{3}$$

$$K = \mu a k,\quad M = \rho a^3 m$$

the equations of motion (1) and (2) reduce to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + Kn(v - u) \tag{4}$$

$$\frac{\partial v}{\partial t} = \frac{k}{m}(u - v) \tag{5}$$

3. Steady State Motion

In steady state, we notice that the dust and fluid particles will flow with same velocity u which is given by

$$\frac{\partial^2 u}{\partial y^2} = 0 \tag{6}$$

together with

$$u(y = -1) = 0 \text{ and } u(y = 1) = u_0 \tag{7}$$

Hence the steady state velocity is given by

$$u = v = \frac{u_0}{2}(1 + y) \tag{8}$$

where u_0 is the non-dimensional velocity of the moving plate.

4. Flow when the Moving Plate is Impulsively Brought to Rest

The fluid velocity and dust velocity $u(y, t)$ and $v(y, t)$ satisfy the differential equations (4) and (5) together with the boundary conditions

$$u(\pm 1, t) = 0 \tag{9}$$

and the initial conditions

$$u(y, 0) = \frac{u_0}{2}(1 + y) \tag{10}$$

$$v(y, 0) = \frac{u_0}{2}(1 + y) \tag{11}$$

The boundary condition on the dust velocity could not be prescribed as dust particles could slip on the boundaries.

Let $\bar{u}$ and $\bar{v}$ be the Laplace Transforms of u and v defined by

$$\bar{u} = \int_0^\infty e^{-st} u(y, t)\, dt$$

and

$$\bar{v} = \int_0^\infty e^{-st} v(y, t)\, dt \tag{12}$$

The Eqns. (4) and (5) reduce to

$$\frac{\partial^2 \bar{u}}{\partial y^2} - (Kn + s)\,\bar{u} + Kn\bar{v} = -\frac{u_0}{2}(1 + y) \tag{13}$$

and

$$\bar{v} = \frac{\bar{u}}{1 + \tau s} + \frac{u_0 \tau (1 + y)}{2(1 + \tau s)} \tag{14}$$

from which we obtain the following differential equation satisfied by $\bar{u}$ as

$$\frac{\partial^2 \bar{u}}{\partial y^2} - \bar{u}A = -(1+y)B \tag{15}$$

where

$$A = s\left(1 + \frac{Kn\tau}{1+s\tau}\right) \tag{16a}$$

and

$$B = \frac{u_0}{2}\left(1 + \frac{Kn\tau}{1+s\tau}\right) \tag{16b}$$

The solution of the equation satisfying the condition of no-slip on the boundary can be obtained as

$$\bar{u}(y, s) = -\frac{u_0}{2s}\left\{\frac{\cosh(\sqrt{A}y)}{\cosh(\sqrt{A})} + \frac{\sinh(\sqrt{A}y)}{\sinh(\sqrt{A})}\right\} + \frac{u_0}{2s}(1+y) \tag{17}$$

Employing this in Eqn. (14), we realize the transform of the dust velocity.

5. Flow Rates

If Q and Q_d be the non-dimensional fluxes or flow rates of fluid and dust particles respectively, then their Laplace transforms are given by

$$\bar{Q} = \int_{-1}^{1} \bar{u}\, dy$$

$$= \frac{u_0}{s\sqrt{A}} \tanh(\sqrt{A}) + \frac{u_0}{s} \tag{18}$$

and

$$\bar{Q}_d = \int_{-1}^{1} \bar{v}\, dy$$

$$= \frac{u_0}{\sqrt{A}(1+s\tau)\, s} \tanh(\sqrt{A}) + \frac{u_0}{s} \tag{19}$$

6. Skin Friction

The non-dimensional skin-friction $\tau_{xy} = \partial u/\partial y$ has the Laplace transform

$$\bar{\tau}_{xy} = \frac{\partial \bar{u}}{\partial y} \text{ on } y = \pm 1$$

Hence

$$\bar{\tau}_{xy} = -\frac{u_0\sqrt{A}}{2s}\{\coth(\sqrt{A}) - \tanh(\sqrt{A})\} + \frac{u_0}{2s} \text{ at } y = -1 \tag{20}$$

$$\bar{\tau}_{xy} = -\frac{u_0\sqrt{A}}{2s}\{\coth(\sqrt{A}) + \tanh(\sqrt{A})\} + \frac{u_0}{2s} \text{ at } y = 1 \tag{21}$$

It can be remarked that the skin-friction on the two plates differ significantly due to the appearance of slip-velocity on the boundary to which impulse is applied.

Exact analytical expressions for the fluid and dust velocities, the flow rates and skin-friction can be obtained by taking the Laplace inverse transforms of the Eqns. (14), (17), (18), (19), (20) and (21). But the computation involved in this process is a bit more complex. However, to understand the physical process, all this may not be necessary. As it is true in the consideration of wave propagation and accoustic propagation the solutions for very short and long times would be more helpful in this context. When the time elapsed from the start of the motion is short compared with the characteristic time τ, the terms involving the partial derivatives with respect to time, t dominate and flow tends to behave as if the dust particles are absent (i.e. clear Newtonian fluid). When the time is long compared with τ, the other terms in the equation dominate and the boundary layer develops as if both fluid and dust move with same velocity.

7. Case 1

When s is considered to be large i.e. at initial times

$$A \approx s + Kn \tag{22}$$

On taking the Laplace inverse transforms of Eqns. (14), (17), (18), (19), (20) and (21) with the help of (22), we get

$$u(y, t) = -\frac{u_0\pi}{2}\left[\sum_{g=1}^{\infty}(-1)^{(g-1)}(2g-1)\cos\left\{\frac{(2g-1)\pi y}{2}\right\}\right.$$

$$\times\{1 - e^{-f(g)t}\} + 2\sum_{g=1}^{\infty}(-1)^g\frac{g\sin(g\pi y)}{h(g)}$$

$$\left.\times\{1 - e^{-h(g)t}\}\right] + \frac{u_0}{2}(1 + y)$$

where

$$h(g) = Kn + \pi^2 g \text{ and } f(g) = \frac{(2g-1)^2\pi^2}{4} + kn$$

Also

$$v(y, t) = -\frac{u_0\pi}{2}\left[\sum_{g=1}^{\infty}(-1)^{(g-1)}\frac{(2g-1)}{f(g)}\cos\left\{\frac{(2g-1)\pi y}{2}\right\}\right.$$

$$\times\left\{(1 - e^{-t/\tau}) + \frac{e^{-f(g)t} - e^{-t/\tau}}{(f(y) - 1}\right\}$$

(equation continued on p. 236)

$$+ 2 \sum_{g=1}^{\infty} (-1)^g \frac{g \sin (g\pi y)}{h(g)} \left\{(1 - e^{-t/\tau}) + \frac{e^{-h(g)t} - e^{-t/\tau}}{h(g) - 1}\right\}\Bigg] + \frac{u_0}{2} (1 + y)$$

$$Q = - \frac{u_0}{\sqrt{kn}} \, Erf \, (\sqrt{knt}) + u_0$$

$$Q_d = u_0 \left[\left\{e^{-t/\tau} \frac{Erf\,(\sqrt{(kn - 1/\tau)\, t}}{\sqrt{kn - 1/\tau}} - \frac{Erf\,(\sqrt{knt})}{\sqrt{kn}}\right\} + 1\right]$$

$$\tau_{xy} = \frac{u_0}{2} \quad \text{at} \quad y = -1$$

and

$$\tau_{xy} = \frac{u_0}{2} \left\{1 - \frac{2}{\sqrt{\pi t}}\right\} \quad \text{at } y = 1$$

In obtaining the expressions for Q and Q_d, the tables[7] are adopted.

Case 2

When s is considered to be small i.e. at large times

$$A \approx s\,(1 + mn) \tag{23}$$

On taking the Laplace inverse transforms of Eqns. (14), (17), (18), (19), (20) and (21) with the help of (22), we obtain

$$u(y, t) = - \frac{u_0}{\pi} \left[2 \sum_{g=1}^{\infty} \frac{(-1)^g}{(2g - 1)} e^{-c(g)t} \cos \left\{\frac{(2g - 1)\, \pi y}{2}\right\} + \sum_{g=1}^{\infty} \frac{(-1)^g}{g} e^{-d(g)t} \sin (g\pi y)\right]$$

where

$$c(g) = \frac{(2g - 1)\, \pi^2}{4(1 + kn\tau)} \quad \text{and} \quad d(g) = \frac{g^2\pi^2}{1 + kn\tau}$$

Also

$$v(y, t) = \frac{u_0}{\pi} \Bigg[2 \sum_{g=1}^{\infty} \frac{(-1)^g}{(2g - 1)} \frac{\cos \left\{\frac{(2g - 1)\, \pi y}{2}\right\}}{1 - \tau c(g)} \times \{e^{-t/\tau} - e^{-c(g)t}\}$$

$$+\sum_{g=1}^{\infty}\frac{(-1)^g}{g}\,\frac{\sin(g\pi y)}{(1-\tau d(g))}\,\{e^{-t/\tau}-e^{-d(g)t}\}\Bigg]$$

$$+\frac{u_0}{2}\,(1+y)\,e^{-t/\tau}$$

$$Q = u_0\left\{1-\frac{8}{\pi^2}\sum_{g=1}^{\infty}\frac{1-e^{-c(g)t}}{(2g-1)^2}\right\}$$

$$Q_d = u_0\left\{1-\frac{8}{\pi^2}\,(1-e^{-t/\tau})\sum_{g=1}^{\infty}\frac{1}{(2g-1)^2}\right.$$

$$\left.-\frac{8}{\pi^2}\sum_{g=1}^{\infty}\frac{1}{(2g-1)^2}\,\frac{e^{-t/\tau}-e^{-c(g)t}}{1-\tau c(g)}\right\}$$

$$\tau_{xy} = -\frac{u_0}{2}\left\{\theta_1\left(\tfrac{1}{2}\,\Big|\,\frac{i\pi t}{(1+kn\tau)}\right)+\theta_4\left(\tfrac{1}{2}\,\Big|\,\frac{i\pi t}{(1+kn\tau)}\right)\right\}+\frac{u_0}{2}$$

$$\text{at}\quad y=-1$$

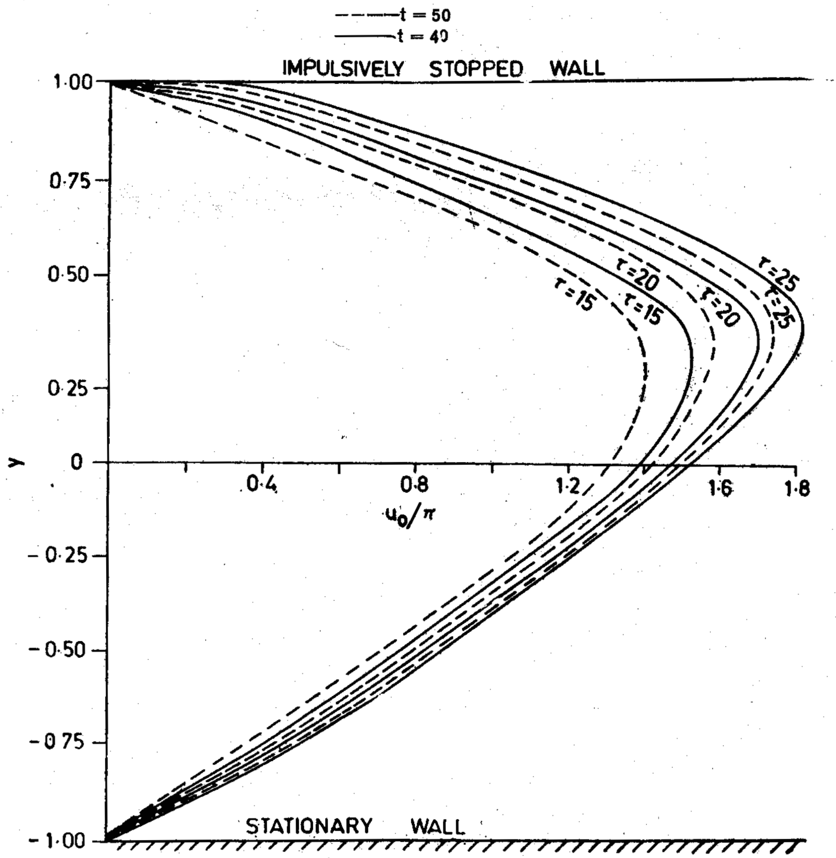

Figure 1. Fluid velocity profiles for $Kn = 20$.

and

$$\tau_{xy} = -\frac{u_0}{2}\left\{-\theta_1\left(\tfrac{1}{2}\,\middle|\,\frac{i\pi t}{(1+kn\tau)}\right)+\theta_4\left(\tfrac{1}{2}\,\middle|\,\frac{i\pi t}{(1+kn\tau)}\right)\right\}+\frac{u_0}{2}$$

$$\text{at} \quad y = 1$$

adopting the formulas No. 34 and 36, page 258, Harry Bateman's[10] project, Table of integral transforms, Vol. I

8. Discussion

When the dust is fine, the relaxation time of dust particles decreases and ultimately as $\tau \to 0$, the velocity of the dusty fluid becomes that of clear fluid in both cases. We have observed that velocities of fluid and dust particles and fluxes of fluid and dust are decreasing exponentially with time and decreasing with number density n, Stokes resistance coefficient k and relaxation time of dust particles τ.

The initial velocities of both the fluid and dust are the same on the plate $y = 1$ whose uniform motion is impulsively stopped. This may be attributed to the fact that the sudden change in the velocities of one of the plates will not make an

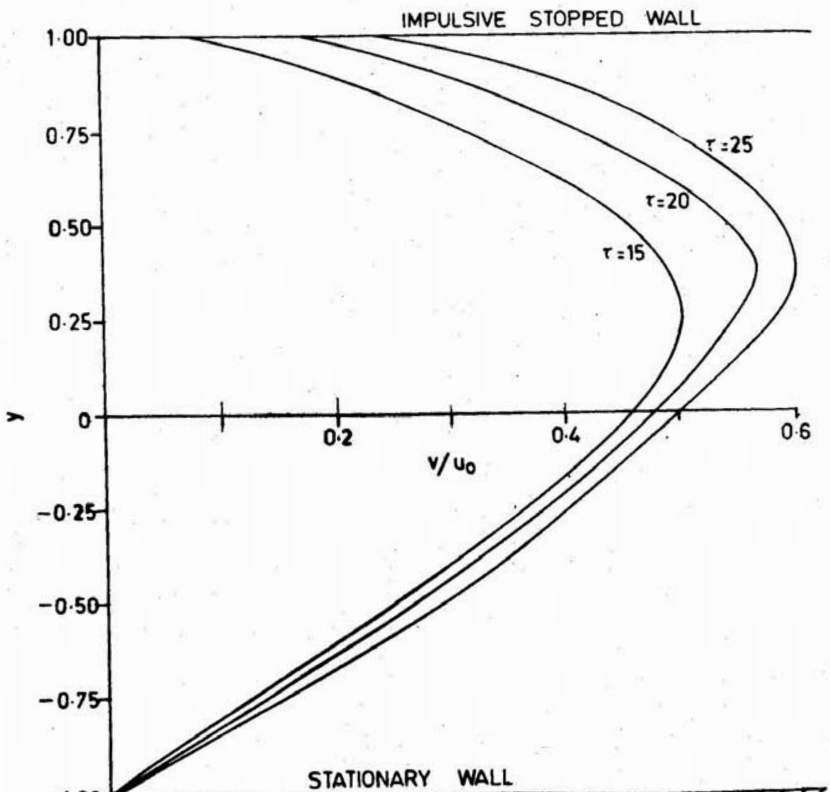

Figure 2. Dust velocity profile for $Kn = 20$ at $t = 50$.

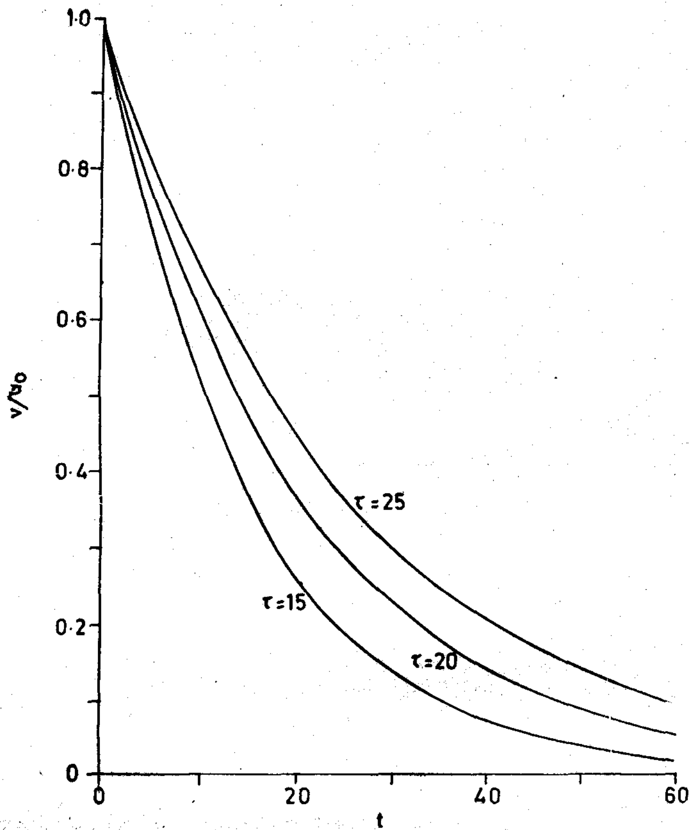

Figure 3. Slip velocity of dust for $Kn = 20$ vs time.

immediate impact on the motion of both the fluid and dust. As time rolls the dust particles on the plate $y = 1$ will not stick to the wall and in fact, slip with the velocity $v = v_0 e^{-t/\tau}$ which exponentially decays with the characteristic time τ, which is the same as the relaxation time.

Figures 1 and 2 show the velocity profiles of fluid and dust at a particular concentration ($kn = 20$) of dust particles with different τ. It can be noticed from Fig. 1 that fluid velocity profiles become flat for small values of τ and large values of t. The maximum velocity in the flow region increases with the increase of t and τ. Further the point at which this maximum is attained is shifted more towards the wall with impulsive motion as t and τ increase. From Fig. 2 the dust velocity profiles exhibit a similar character. The variation of the slip-velocity of dust particles with time and τ is illustrated in Fig. 3. It is noticed that slip decreases with increasing t and decreasing τ.

References

1. Saffman, P. G., *J. Fluid Mechanics*, **13** (1962), 120.
2. Michael, D. H. & Miller, D. A., *Mathemetica*, **13** (1966), 97.
3. Michael, D. H. & Norey, S. W., *Quart. J. Mech. and Appl. Maths*, **21**, (1968), 375.
4. Nath, G., *Proc. Nat. Ac. Sc.* **40** (1970), 25.

5. Mukharjee, S. R., Tiwari, V. D. & Bhattacharjee, S., *Progress of Mathematics*, **7**, (1973) **37**.
6. Gupta, R. K. & Gupta, S. C., *J. Appld. Maths–Phys.* (*Zamp*), **27** (1976), 119–25.
7. Edward, W. Ng. & Murray Geller, *J. of Research of the National Bureau of standards-B. Mathematical Sciences*. **73B** (Jan-Mar. 1969).
8. Marble, E. Frank, *Ann. Rev. Fluid Mech.*, **2** (1970), 397.
9. Cox, R. G. & Mason., S. G., *Ann. Rev. Fluid Mech.*, **3** (1971), 291.
10. Harry Bateman, Manuscript project, *Table of Integral Transforms*, **1**, (1954), 258.
11. Liu, J. T. G., *Phys. Fluid*, **9** (1966), 1916–26.
12. Liu, J. T. C., *Astronautica Acta.*, **13** (1967), 369–77.
13. Verma, P. D. & Mathur, A. K., *Indian J. Pure & Applied Maths.*, **14** (1973), 133.
14. Ramana Prasad, V. V. & Pattabhi Ramacharyulu, N. Ch., *Def. Sci. J.* **30**, (1980), 125.