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Secondary Flow and Heat Transfer of Two Incompressible Immiscible Fluids Between Two Parallel Plates in a Rotating System

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Abstract. When a straight channel formed by two parallel plates, through which two immiscible fluids are flowing under constant pressure gradients is rotated about an axis perpendicular to the plates, secondary motion is set up. The secondary motion is analysed in detail for constant angular velocity Ω', λ the ratio of the densities of the lower and upper fluids and μ^2 the corresponding ratio of their viscosities. The associated heat transfer problems, when the plate temperatures are equal and different, have also been studied.

1. Introduction

The fundamental difficulty in solving the Navier-Stokes equations either exactly or approximately is the non-linearity introduced by the convection terms in the momentum equations. There exist, however, non-trivial problems in which the convection terms vanish and these provide the simple class of solutions of the equations of motion. One such flow has been considered recently by Vidyanidhi & Nigam[1] who have studied the secondary flow when a straight channel, formed by two parallel plates through which fluid is flowing under a constant pressure gradient, is rotated about an axis perpendicular to the plates. This problem was later extended by Vidyanidhi[2] in the frame-work of hydromagnetics and by Vidyanidhi, Bala Prasad & Ramana Rao[3] to include the effects of uniform suction and injection. The latter analysis has been made use of by Ramana Rao & Bala Prasad[4] in studying the temperature distribution. The influence of stratification on rotating fluids was brought about by Niimi[5] for the flow between two parallel infinite disks. The hydrodynamic couette flow and heat transfer in a rotating frame of reference was also studied by Jana & Datta[6] and later extended in the frame-work of hydromagnetics by Jana, Datta & Mazumder[7]. These problems have wide applications in designing thermo-syphon tube, in cooling turbine blades and have some bearing in MHD power generation.

The velocity profile due to the flow of two incompressible immiscible fluids between two parallel plates and occupying equal heights was obtained by Bird, Stewart & Lightfoot[8]. This problem was further generalised by Kapur & Shukla[9] to the case of flow of a number of incompressible immiscible fluids occupying different heights. The stability analysis of two superposed fluids between parallel planes was formulated by Yih[10] and later extended by Nakaya & Hasegawa[11] to include the effects of gravity and surface tension.

The authors[12] have recently extended the work of Jana & Datta[6] for the flow of two incompressible immiscible fluids, occupying equal heights between two parallel plates. The present paper is an extension of the work of Vidyanidhi & Nigam[1], in which heat transfer characteristics, assuming equal and different plate temperatures have also been studied. Olive-oil and water can be taken as the two immiscible fluids to test the theoretical conclusions of this work for setting up an experiment as suggested by Vidyanidhi & Nigam[1].

2. The Basic Equations and Solutions

The equations of motion and continuity for the steady state in a rotating frame of reference $0'X'Y'Z'$ as in Squire[13] for two incompressible and immiscible fluids as shown in Fig. 1 are,

$$(\vec{U}'_m \cdot \vec{\nabla}')\, \vec{U}'_m + 2\vec{\Omega}' x \vec{U}'_m = -\rho_m^{-1}\, \vec{\nabla}' \pi'_m + \nu_m \vec{\nabla}^2 \vec{U}'_m \tag{1}$$

$$\vec{\nabla}' \cdot \vec{U}'_m = 0 \tag{2}$$

$$\pi'_m \text{ (the modified pressures)} = p_m - \tfrac{1}{2}\rho_m \,|\, \vec{\Omega}' x \vec{r}' \,|^2, \ (m = 1, 2) \tag{3}$$

Here the subscripts 1 and 2 refer to the upper and lower fluids in the ranges $0 \leqslant z' \leqslant L$ and $-L \leqslant z' \leqslant 0$ respectively. $\vec{U}'_1$, $\vec{U}'_2$, $\vec{\Omega}'$ and $\vec{r}'$ are the velocities of the upper fluid, lower fluid, angular velocity and position vector respectively.

We choose a right handed cartesian system such that z'-axis is perpendicular to the motion of the fluids under the action of constant pressure gradients $P'_m = (-\partial\pi'_m/\partial x')$ in the direction of x'-axis between two parallel plates $z' = \pm L$ (stationary relative to $0'X'Y'Z'$).

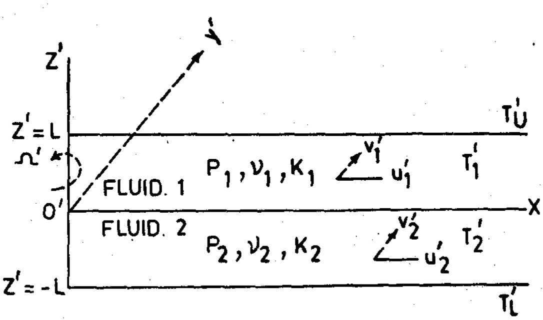

Figure 1. Schematic diagram.

Assuming that π'_m is independent of y' and z', π'_m is given by

$$-\pi'_m = \left(\frac{p'_{m_1} - p'_{m_2}}{D}\right)x' - p'_{m1}, \tag{4}$$

where p'_{m_1} and p'_{m_2} stand for the pressures on the planes $x' = 0$ and $x' = D$ respectively.

The velocities of the two fluids are then represented by

$$\vec{U}'_1 = [u'_1(z'), v'_1(z'), 0], \vec{U}'_2 = [u'_2(z'), v'_2(z'), 0] \tag{5}$$

$$\vec{\Omega}' = (0, 0, \Omega') \tag{6}$$

Introducing the non-dimensional quantities,

$$\vec{r}' = \vec{r}L,\ u'_m = \frac{P'_m L^2}{2\rho_1\nu_1} u_m,\ v'_m = \frac{P'_m L^2}{2\rho_1\nu_1} v_m,\ \Omega' = \frac{\alpha^2\nu_2}{L^2} \tag{7}$$

(Taylor number for the lower fluid), $\rho_2 = \lambda\rho_1$, $\nu_2 = \mu^2\nu_1$, the Eqn. (1) reduces to

$$-2\alpha^2\mu^2 v_1 = 2 + \frac{d^2u_1}{dz^2},\ 2\alpha^2\mu^2 u_1 = \frac{d^2v_1}{dz^2},\ (0 \leqslant z \leqslant 1) \tag{8}$$

$$-2\alpha^2 v_2 = \frac{2}{\lambda\mu^2} + \frac{d^2u_2}{dz^2},\ 2\alpha^2 u_2 = \frac{d^2v_2}{dz^2},\ (-1 \leqslant z \leqslant 0). \tag{9}$$

We seek the solutions of Eqns. (8) and (9) subject to the boundary conditions

$$\left.\begin{aligned} & u_1 = v_1 = 0 \text{ at } z = 1,\ u_1 = u_2,\ v_1 = v_2 \text{ at } z = 0 \\ & \rho_1\nu_1\frac{du_1}{dz} = \rho_2\nu_2\frac{du_2}{dz},\ \rho_1\nu_1\frac{dv_1}{dz} = \rho_2\nu_2\frac{dv_2}{dz} \text{ at } z = 0, \\ & u_2 = v_2 = 0 \text{ at } z = -1. \end{aligned}\right\} \tag{10}$$

In terms of the complex notation,

$$q_1 = u_1 + iv_1,\ \ q_2 = u_2 + iv_2 \tag{11}$$

the Eqns. (8), (9) and (10) become

$$\frac{d^2q_1}{dz^2} - 2i\mu^2\alpha^2 q_1 = -2 \tag{12}$$

$$\frac{d^2q_2}{dz^2} - 2i\alpha^2 q_2 = -\frac{2}{\lambda\mu^2} \tag{13}$$

$$\left.\begin{aligned} & q_1 = 0 \text{ at } z = 1,\ \ q_1 = q_2 \text{ at } z = 0,\ \frac{dq_1}{dz} = \lambda\mu^2\frac{dq_2}{dz} \text{ at } z = 0, \\ & q_2 = 0 \text{ at } z = -1. \end{aligned}\right\} \tag{14}$$

Solving Eqns. (12) and (13) subject to Eqn. (14), we get

$$q_1 = \frac{i}{\Delta \mu^2 \alpha^2} [\{\text{Sh}(1+i)\alpha + \mu [\text{Sh}(1+i)\alpha\mu + (\lambda - 1)\text{Ch}(1+i)\alpha$$

$$\text{Sh}(1+i)\alpha\mu]\}\text{Ch}(1+i)\alpha\mu z + \mu\{\lambda \text{Ch}(1+i)\alpha$$

$$- \text{Ch}(1+i)\alpha\mu - (\lambda - 1)\text{Ch}(1+i)\alpha\,\text{Ch}(1+i)\alpha\mu\}$$

$$\text{Sh}(1+i)\alpha\mu z - \Delta] \tag{15}$$

$$q_2 = \frac{i}{\Delta \mu^2 \alpha^2 \lambda} [\{\lambda\,\text{Sh}(1+i)\alpha + \lambda\mu\,\text{Sh}(1+i)\alpha\mu - (\lambda - 1)$$

$$\text{Sh}(1+i)\alpha\,\text{Ch}(1+i)\alpha\mu\}\text{Ch}(1+i)\alpha z + \{\lambda \text{Ch}(1+i)\alpha$$

$$- \text{Ch}(1+i)\alpha\mu - (\lambda - 1)\text{Ch}(1+i)\alpha\,\text{Ch}(1+i)\alpha\mu\}$$

$$\text{Sh}(1+i)\alpha z - \Delta] \tag{16}$$

$$\Delta = \text{Sh}(1+i)\alpha\,\text{Ch}(1+i)\alpha\mu + \lambda\mu\,\text{Sh}(1+i)\alpha\mu\,\text{Ch}(1+i)\alpha \tag{17}$$

where Sh and Ch stand for hyperbolic sine and hyperbolic cosine respectively.

Separating Eqns. (15) and (16) into real and imaginary parts, we get in terms of the following,

$$a_0 = \lambda\mu + 1,\ b_0 = \lambda\mu - 1,\ a_1 = \alpha(\mu + 1),\ b_1 = \alpha(\mu - 1),$$

$$a_2 = a_0\,\text{Sh}\,a_1 \cos a_1 + b_0\,\text{Sh}\,b_1 \cos b_1,\ b_2 = a_0\,\text{Ch}\,a_1 \sin a_1$$

$$+ b_0\,\text{Ch}\,b_1 \sin b_1,\ k = a_2^2 + b_2^2,\ p = \alpha(\mu z + 1),$$

$$q = \alpha(\mu z - 1),\ s = a_0\,\text{Sh}\,p \cos p + b_0\,\text{Sh}\,q \cos q,$$

$$t = a_0\,\text{Ch}\,p \sin p + b_0\,\text{Ch}\,q \sin q,\ r = \alpha(1 + z),\ f = \text{Sh}\,r \cos r,$$

$$g = \text{Ch}\,r \sin r,\ r_0 = p - a_1,\ f_0 = \text{Sh}\,r_0 \cos r_0,\ g_0 = \text{Ch}\,r_0 \sin r_0,$$

$$f_1 = r_0 + \alpha,\ g_1 = r_0 - \alpha,\ f_2 = \text{Sh}\,f_1 \cos f_1 + \text{Sh}\,g_1 \cos g_1,$$

$$g_2 = \text{Ch}\,f_1 \sin f_1 + \text{Ch}\,g_1 \sin g_1,\ f_3 = s - 2\mu f_0 - \mu(\lambda - 1) f_2 - a_2,$$

$$g_3 = t - 2\mu g_0 - \mu(\lambda - 1) g_2 - b_2,\ f_4 = b_1 + r,\ g_4 = a_1 - r,$$

$$f_5 = b_0\,\text{Sh}\,f_4 \cos f_4 + a_0\,\text{Sh}\,g_4 \cos g_4,\ g_5 = b_0\,\text{Ch}\,f_4 \sin f_4$$

$$+ a_0\,\text{Ch}\,g_4 \sin g_4,\ f_6 = \alpha\mu + r,\ g_6 = \alpha\mu - r,$$

$$f_7 = \text{Sh}\,f_6 \cos f_6 - \text{Sh}\,g_6 \cos g_6,\ g_7 = \text{Ch}\,f_6 \sin f_6 - \text{Ch}\,g_6 \sin g_6,$$

$$f_8 = 2\lambda f + f_5 - a_2 - (\lambda - 1) f_7,\ g_8 = 2\lambda g + g_5 - b_2 - (\lambda - 1) g_7 \tag{18}$$

$$u_1 = \frac{b_2 f_3 - a_2 g_3}{k\mu^2\alpha^2},\quad v_1 = \frac{a_2 f_3 + b_2 g_3}{k\mu^2\alpha^2} \tag{19}$$

$$u_2 = \frac{b_2 f_8 - a_2 g_8}{\lambda k\mu^2\alpha^2},\quad v_2 = \frac{a_2 f_8 + b_2 g_8}{\lambda k\mu^2\alpha^2} \tag{20}$$

As $\alpha \to 0$,

$$u_1 = \frac{1}{\lambda\mu^2 + 1}\,[2 + (\lambda\mu^2 - 1)\,z - (\lambda\mu^2 + 1)\,z^2],\ v_1 = 0, \tag{21}$$

$$\left.\begin{aligned} u_2 &= \frac{1}{\lambda\mu^2(\lambda\mu^2 + 1)}\,[2\lambda\mu^2 + (\lambda\mu^2 - 1)\,z - (\lambda\mu^2 + 1)\,z^2]\} \\ v_2 &= 0 \end{aligned}\right\} \tag{22}$$

The skin-friction amplitudes at the upper and lower plates are respectively given by

$$\Gamma_U = \frac{1}{\mu\alpha}\sqrt{\frac{2(c_4^2 + d_4^2)}{k}},\ \Gamma_L = \frac{1}{\lambda\mu^2\alpha}\sqrt{\frac{2(c_5^2 + d_5^2)}{k}}, \tag{23}$$

where

$$\begin{aligned} c_4 &= a_0 \operatorname{Ch} a_1 \cos a_1 + b_0 \operatorname{Ch} b_1 \cos b_1 - 2\mu\,(\lambda - 1) \operatorname{Ch} \alpha \cos \alpha - 2\mu, \\ d_4 &= -\,a_0 \operatorname{Sh} a_1 \sin a_1 - b_0 \operatorname{Sh} b_1 \sin b_1 + 2\mu\,(\lambda - 1) \operatorname{Sh} \alpha \sin \alpha, \\ c_5 &= -\,2\lambda - b_0 \operatorname{Ch} b_1 \cos b_1 + a_0 \operatorname{Ch} a_1 \cos a_1 + 2\,(\lambda - 1) \operatorname{Ch} \mu\alpha \cos \mu\alpha, \\ d_5 &= -\,b_0 \operatorname{Sh} b_1 \sin b_1 + a_0 \operatorname{Sh} a_1 \sin a_1 + 2\,(\lambda - 1) \operatorname{Sh} \mu\alpha \sin \mu\alpha. \end{aligned} \tag{24}$$

As $\alpha \to 0$,

$$\Gamma_U = \frac{\lambda\mu^2 + 3}{\lambda\mu^2 + 1}, \quad \Gamma_L = \frac{1 + 3\lambda\mu^2}{\lambda\mu^2\,(\lambda\mu^2 + 1)} \tag{25}$$

For large Ω' such that $(P_1'/\mu^2\alpha^2)$ remains finite, we obtain from Eqn. (15) for $1 \geqslant z \geqslant 0$,

$$\left.\begin{aligned} u_1 &\simeq \frac{1}{\mu^2\alpha^2}\,e^{\alpha\mu(z-1)} \sin \{\alpha\mu\,(1 - z)\}, \\ v_1 &\simeq \frac{1}{\mu^2\alpha^2}\,[-1 + e^{\alpha\mu(z-1)} \cos \{\alpha\mu\,(1 - z)\}], \end{aligned}\right\} \tag{26}$$

and for $0 \geqslant z \geqslant -1$,

$$\left.\begin{aligned} u_2 &\simeq \frac{1}{\lambda\mu^2\alpha^2}\,e^{-\alpha(1+z)} \sin \alpha\,(1 + z), \\ v_2 &\simeq \frac{1}{\lambda\mu^2\alpha^2}\,[-1 + e^{-\alpha(1+z)} \cos \alpha\,(1 + z)], \end{aligned}\right\} \tag{27}$$

when $(P_2'/\lambda\mu^2\alpha^2)$ remains finite.

3. Results and Discussions

The velocity distributions for the primary and secondary flows have been shown in Figs. 2 and 3 to illustrate the effects of the various parameters λ, μ and α. It is concluded that as λ increases, the primary flow decreases and the secondary flow

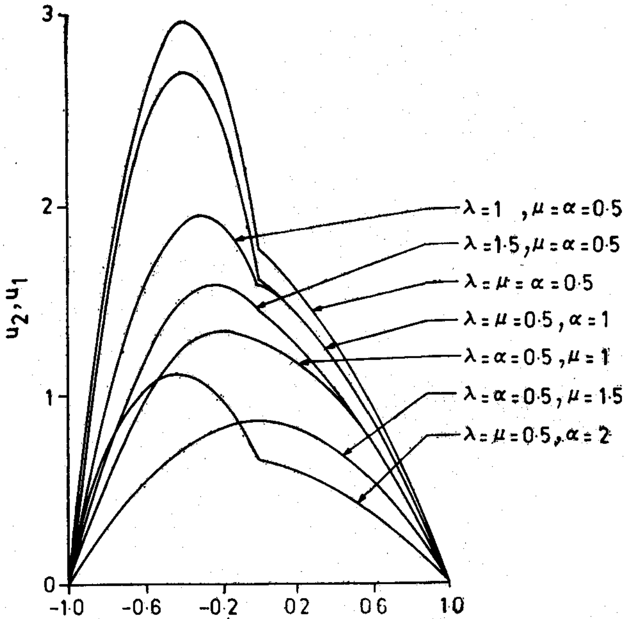

Figure 2. Velocity profiles of the primary flow.

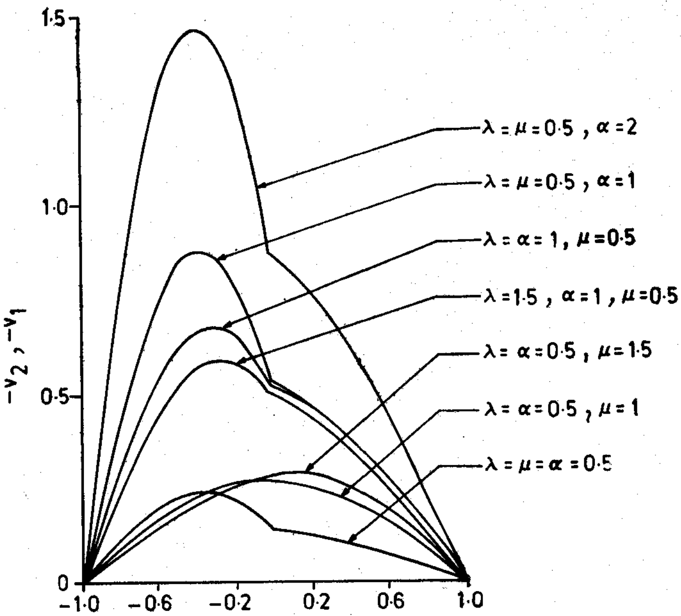

Figure 3. Velocity profiles of the secondary flow.

increases at any point of the channel. As μ increases, the primary flow always decreases whereas the secondary flow decreases for the upper fluid and increases close to the lower plate. Also it is noted that as α increases the primary flow decreases and the secondary flow decreases or increases in magnitude at any point of the channel. We note from Eqn. (26) that the amplitude of u_1 is positive and that the function $\sin \alpha\mu(1 - z)$ can take positive or negative values. For $\alpha \to \infty$ such that (i) when $(P_1'/\mu^2\alpha^2)$ is finite, the secondary flow is confined to regions of order $(L/\alpha\mu)$ in the vicinity of the upper plate, the thickness of the boundary layer being of order $(\Omega'/\nu_1)^{-1/2}$, (ii) when $(P_2'/\lambda\mu^2\alpha^2)$ is finite, the secondary flow is confined to regions of order (L/α) in the vicinity of the lower plate, the thickness of the boundary layer being of order $(\Omega'/\nu_2)^{-1/2}$.

The skin-friction amplitudes at the upper and lower plates have been shown in Figs. 4 and 5 respectively for various values of λ, μ and α. It is concluded that the skin-friction amplitude at either plate decreases with an increase in any of the parameters.

It may be possible to perform experiments by rotating a channel of finite width B which is large compared with the depth $2L$. In such a channel the conditions close to the walls $z' = \pm L$ are not given by the above calculations but if the side walls are in such a direction that there is no total flow across them, then the conditions can be attained approximately over most of the channel. It is necessary to keep the side walls at an angle '$-\phi$' with x'-axis where

$$\tan \phi = \frac{\int_{-1}^{0} v_2 dz + \int_{0}^{1} v_1 dz}{\int_{-1}^{0} u_2 dz + \int_{0}^{1} u_1 dz} \tag{28}$$

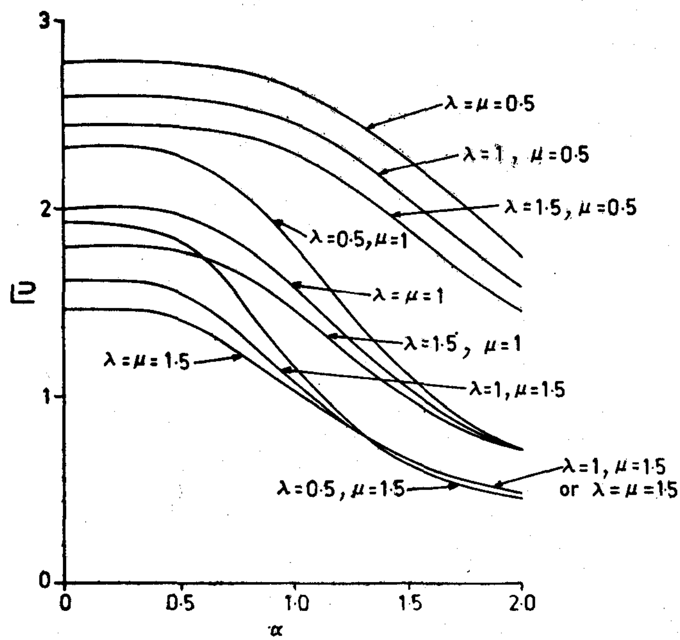

Figure 4. Skin friction amplitude Γ_U at the upper plate.

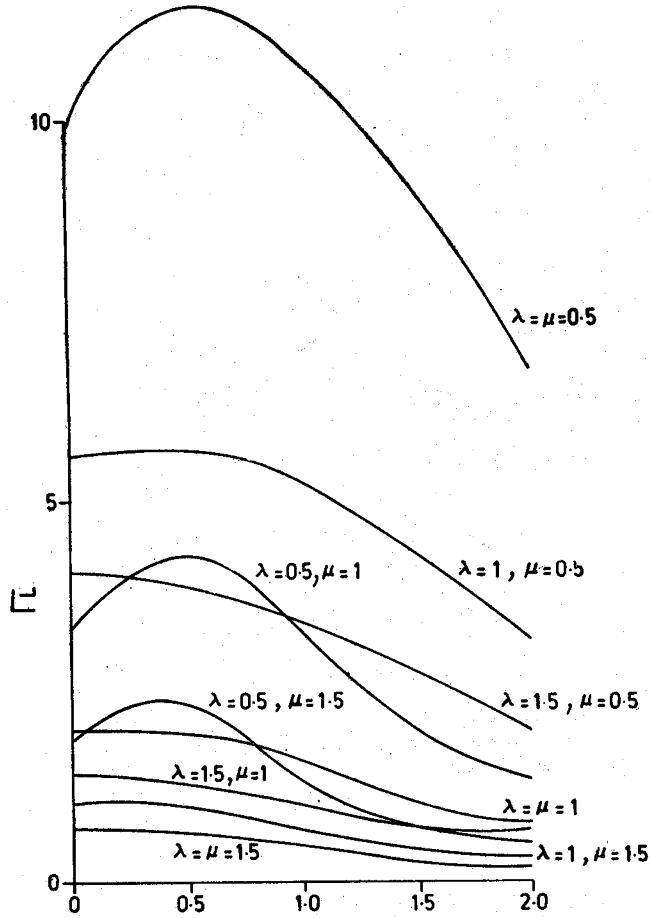

Figure 5. Skin friction amplitude Γ_L at the lower plate.

It is concluded that the angle ϕ for any α decreases as λ or μ increases. In other words the effect of λ or μ is to inhibit the secondary flow through the side walls.

4. Heat Transfer

The temperature fields for the upper and lower fluids can now be determined from the heat transfer equations

$$0 = \frac{\nu_m}{C_p} \frac{dq'_m}{dz'} \frac{d\bar{q}'_m}{dz'} + K_m \frac{d^2T'_m}{dz'^2} \tag{29}$$

for $m = 1$ and 2 respectively. Here $q_m = u'_m + iv'_m$, $\bar{q}'_m = u'_m - iv'_m$, K_1 and K_2 are the thermal diffusivities of the upper and lower fluids respectively. C_p is the specific heat at constant pressure. The first term on the R.H.S. of the above equation is due to the viscous dissipation.

Plates Maintained at Equal Temperatures

It is assumed that the temperature of either plate is a constant T'_L. Introducing the non-dimensional quantities of Eqn. (7) and in addition,

$$\left.\begin{aligned} &\theta_1' = (T'_1 - T'_L)/T'_L,\ \theta_2' = (T'_2 - T'_L)/T'_L,\\ &P_1 \text{ (Prandtl number for the upper fluid)} = \nu_1/K_1,\\ &E_c \text{ (Eckert number for the upper fluid)} = \left(\frac{P_1' L^2}{2\rho_1\nu_1}\right)^2 \Big/ C_p T'_L,\\ &\eta = K_2/K_1, \end{aligned}\right\} \tag{30}$$

we obtain from Eqn. (29),

$$\frac{d^2\theta_1'}{dz^2} = -P_1 E_c \frac{dq_1}{dz}\frac{d\bar{q}_1}{dz},\quad \frac{d^2\theta_2'}{dz^2} = -\frac{\mu^2}{\eta} P_1 E_c \frac{dq_2}{dz}\frac{d\bar{q}_2}{dz} \tag{31}$$

as the non-dimensional temperature equations for the upper and lower fluids respectively.

Solving Eqn. (31) subject to the boundary conditions in non-dimensional form,

$$\theta_1' = 0 \text{ at } z = 1,\ \theta_1' = \theta_2' \text{ at } z = 0,\ \frac{d\theta_1'}{dz} = \eta\frac{d\theta_2'}{dz} \text{ at } z = 0;\ \theta_2' = 0 \text{ at } z = -1 \tag{32}$$

we get in terms of the following constants,

$$a_4 = (1 + \lambda^2\mu^2)\,\text{Ch}\,2\alpha - (1 - \lambda^2\mu^2)\cos 2\alpha,$$

$$b_4 = (1 - \lambda^2\mu^2)\,\text{Ch}\,2\alpha - (1 + \lambda^2\mu^2)\cos 2\alpha,$$

$$a_5 = 2\lambda\mu\,\text{Sh}\,2\alpha,\ b_5 = 2\lambda\mu\sin 2\alpha,\ a_6 = a_4\,\text{Ch}\,2\mu\alpha + b_4\cos 2\mu\alpha + a_5\,\text{Sh}\,2\mu\alpha + b_5\sin 2\mu\alpha,$$

$$\begin{aligned} A_1 = \tfrac{1}{2}\Bigg[&a_4 + \mu^2\,\text{Ch}\,2\mu\alpha\left[2 + 4(\lambda - 1)\,\text{Ch}\,\alpha\cos\alpha + \frac{(\lambda-1)^2(a_4-b_4)}{2\lambda^2\mu^2}\right]\\ &+ (2\mu - 2\mu^2\lambda)\,\text{Ch}\,a_1\cos b_1 - (2\mu + 2\mu^2\lambda)\,\text{Ch}\,b_1\cos a_1\\ &+ \left(1 - \frac{1}{\lambda}\right)[a_5\,\text{Sh}\,\alpha\mu\cos\alpha\mu + b_5\,\text{Ch}\,\alpha\mu\sin\alpha\mu\\ &- (a_4 - b_4)\,\text{Ch}\,\alpha\mu\cos\alpha\mu]\Bigg], \end{aligned}$$

$$B_1 = \tfrac{1}{2}\Bigg[b_4 - \mu^2\cos 2\mu\alpha\Bigg[2 + 4(\lambda - 1)\,\text{Ch}\,\alpha\cos\alpha$$

(equation continued on p. 190)

$$+ \frac{(\lambda - 1)^2 (a_4 - b_4)}{2\lambda^2\mu^2}\Big] + (2\mu + 2\mu^2\lambda) \text{ Ch } a_1 \cos b_1$$

$$- (2\mu - 2\mu^2\lambda) \text{ Ch } b_1 \cos a_1 + \left(1 - \frac{1}{\lambda}\right) [a_5 \text{ Sh } \alpha\mu \cos \alpha\mu$$

$$+ b_5 \text{ Ch } \alpha\mu \sin \alpha\mu + (a_4 - b_4) \text{ Ch } \alpha\mu \cos \alpha\mu] \Big],$$

$$A_2 = \tfrac{1}{2} \Big[a_5 + 2\mu (\mu\lambda - 1) \text{ Sh } a_1 \cos b_1 + 2\mu (\mu\lambda + 1) \text{ Sh } b_1 \cos a_1$$

$$- 2\mu^2 \text{ Sh } 2\mu\alpha \left[1 + 2(\lambda - 1) \text{ Ch } \alpha \cos \alpha + \frac{(\lambda - 1)^2 (a_4 - b_4)}{4\lambda^2\mu^2} \right]$$

$$+ \left(1 - \frac{1}{\lambda}\right) [(a_4 - b_4) \text{ Sh } \alpha\mu \cos \alpha\mu - a_5 \text{ Ch } \alpha\mu \cos \alpha\mu$$

$$- b_5 \text{ Sh } \alpha\mu \sin \alpha\mu] \Big],$$

$$B_2 = \tfrac{1}{2} \Big[b_5 + 2\mu (\mu\lambda - 1) \text{ Ch } b_1 \sin a_1 + 2\mu (\mu\lambda + 1) \text{ Ch } a_1 \sin b_1$$

$$- 2\mu^2 \sin 2\mu\alpha \left[1 + 2(\lambda - 1) \text{ Ch } \alpha \cos \alpha + \frac{(\lambda - 1)^2 (a_4 - b_4)}{4\lambda^2\mu^2} \right]$$

$$+ \left(1 - \frac{1}{\lambda}\right) [(a_4 - b_4) \text{ Ch } \alpha\mu \sin \alpha\mu + a_5 \text{ Sh } \alpha\mu \sin \alpha\mu$$

$$- b_5 \text{ Ch } \alpha\mu \cos \alpha\mu] \Big],$$

$$A_4 = \tfrac{1}{2} [2\lambda^2 \text{ Ch } 2\alpha + (\lambda^2\mu^2 + 1) \text{ Ch } 2\mu\alpha - (\lambda^2\mu^2 - 1) \cos 2\mu\alpha$$

$$+ 2\lambda (\lambda\mu - 1) \text{ Ch } a_1 \cos b_1 - 2\lambda (\lambda\mu + 1) \text{ Ch } b_1 \cos a_1$$

$$- 4\lambda (\lambda - 1) \text{ Ch } 2\alpha \text{ Ch } \alpha\mu \cos \alpha\mu + 2(\lambda - 1) [(\text{Ch } 2\alpha\mu$$

$$+ \cos 2\alpha\mu) \text{ Ch } \alpha \cos \alpha - \lambda\mu \text{ Sh } 2\alpha\mu \text{ Sh } \alpha \cos \alpha$$

$$- \lambda\mu \sin 2\alpha\mu \text{ Ch } \alpha \sin \alpha] + (\lambda - 1)^2 \text{ Ch } 2\alpha (\text{Ch } 2\alpha\mu + \cos 2\alpha\mu)],$$

$$B_4 = \tfrac{1}{2} [-2\lambda^2 \cos 2\alpha + (\lambda^2\mu^2 - 1) \text{ Ch } 2\mu\alpha - (\lambda^2\mu^2 + 1) \cos 2\mu\alpha$$

$$+ 2\lambda (\lambda\mu + 1) \text{ Ch } a_1 \cos b_1 - 2\lambda (\lambda\mu - 1) \text{ Ch } b_1 \cos a_1$$

$$+ 4\lambda (\lambda - 1) \cos 2\alpha \text{ Ch } \alpha\mu \cos \alpha\mu - 2(\lambda - 1) [\lambda\mu \text{ Sh } 2\alpha\mu \text{ Sh } \alpha \cos \alpha$$

$$+ \lambda\mu \sin 2\alpha\mu \text{ Ch } \alpha \sin \alpha + (\text{Ch } 2\alpha\mu + \cos 2\alpha\mu) \text{ Ch } \alpha \cos \alpha]$$

$$- (\lambda - 1)^2 (\text{Ch } 2\alpha\mu + \cos 2\alpha\mu) \cos 2\alpha],$$

$$A_5 = \lambda^2 \text{ Sh } 2\alpha + \lambda^2\mu (\text{Sh } a_1 \cos b_1 + \text{Sh } b_1 \cos a_1)$$

$$- \lambda(\lambda - 1)\, 2 \text{ Sh } 2\alpha \text{ Ch } \alpha\mu \cos \alpha\mu - \lambda\mu \text{ Sh } 2\alpha\mu - \lambda (\text{Sh } a_1 \cos b_1 -$$

(*equation continued on p. 191*)

$$- \text{Sh}\, b_1 \cos a_1) + (\lambda - 1)(\text{Ch}\, 2\alpha\mu + \cos 2\alpha\mu)\, \text{Sh}\, \alpha \cos \alpha$$

$$- \lambda\mu (\lambda - 1)(\text{Sh}\, 2\alpha\mu\, \text{Ch}\, \alpha \cos \alpha + \sin 2\alpha\mu\, \text{Sh}\, \alpha \sin \alpha)$$

$$+ \tfrac{1}{2} (\lambda - 1)^2\, \text{Sh}\, 2\alpha\, (\text{Ch}\, 2\alpha\mu + \cos 2\alpha\mu),$$

$$B_5 = \lambda^2 \sin 2\alpha + \lambda^2\mu\, (\text{Ch}\, a_1 \sin b_1 + \text{Ch}\, b_1 \sin a_1) - 2\lambda (\lambda - 1)$$

$$\sin 2\alpha\, \text{Ch}\, \alpha\mu \cos \alpha\mu - \lambda\mu \sin 2\alpha\mu - \lambda\, (\text{Ch}\, b_1 \sin a_1$$

$$- \text{Ch}\, a_1 \sin b_1) + (\lambda - 1)(\text{Ch}\, 2\alpha\mu + \cos 2\alpha\mu)\, \text{Ch}\, \alpha \sin \alpha$$

$$- \lambda\mu (\lambda - 1)(\sin 2\alpha\mu\, \text{Ch}\, \alpha \cos \alpha - \text{Sh}\, 2\alpha\mu\, \text{Sh}\, \alpha \sin \alpha)$$

$$+ \tfrac{1}{2} (\lambda - 1)^2 \sin 2\alpha\, (\text{Ch}\, 2\alpha\mu + \cos 2\alpha\mu),$$

$$A_7 = A_1\, \text{Ch}\, 2\alpha\mu + B_1 \cos 2\alpha\mu + A_2\, \text{Sh}\, 2\alpha\mu + B_2 \sin 2\alpha\mu,$$

$$B_7 = A_4\, \text{Ch}\, 2\alpha + B_4 \cos 2\alpha - A_5\, \text{Sh}\, 2\alpha - B_5 \sin 2\alpha \tag{33}$$

$$A_3 = \frac{P_1 E_c}{(1 + \eta)\, a_6 \alpha^4 \mu^4 \lambda^2} [\lambda^2 (A_7 + A_1\eta + B_1\eta - 2\mu\alpha A_2 - 2\mu\alpha B_2)$$

$$+ \mu^2 (B_7 - A_4 - B_4 + 2\alpha A_5 + 2\alpha B_5)],$$

$$B_3 = \frac{P_1 E_c}{(1 + \eta)\, a_6 \alpha^4 \mu^4 \lambda^2} [\lambda^2 (\eta A_7 - \eta A_1 - \eta B_1 + 2\mu\alpha A_2 + 2\mu\alpha B_2)$$

$$- \mu^2 (B_7 - A_4 - B_4 + 2\alpha A_5 + 2\alpha B_5)],$$

$$A_6 = \frac{P_1 E_c}{\eta (\eta + 1)\, a_6 \alpha^4 \mu^4 \lambda^2} [\lambda^2 \eta (A_7 - A_1 - B_1 - 2\mu\alpha A_2 - 2\mu\alpha B_2)$$

$$+ \mu^2 (A_4 + B_4 + \eta B_7 + 2\eta\alpha A_5 + 2\eta\alpha B_5)],$$

$$B_6 = \frac{P_1 E_c}{\eta (\eta + 1)\, a_6 \alpha^4 \mu^4 \lambda^2} [\lambda^2 \eta (A_7 - A_1 - B_1 - 2\mu\alpha A_2 - 2\mu\alpha B_2)$$

$$+ \mu^2 (A_4 + B_4 - B_7 + 2\eta\alpha A_5 + 2\eta\alpha B_5)] \tag{34}$$

$$\theta_1' = A_3 + B_3 z - \frac{P_1 E_c}{a_6 \mu^4 \alpha^4} [A_1\, \text{Ch}\, 2\mu\alpha z + B_1 \cos 2\mu\alpha z$$

$$+ A_2\, \text{Sh}\, 2\mu\alpha z + B_2 \sin 2\mu\alpha z] \tag{35}$$

$$\theta_2' = A_6 + B_6 z - \frac{P_1 E_c}{a_6 \mu^2 \lambda^2 \gamma_1 \alpha^4} [A_4\, \text{Ch}\, 2\alpha z + B_4 \cos 2\alpha z + A_5\, \text{Sh}\, 2\alpha z$$

$$+ B_5 \sin 2\alpha z] \tag{36}$$

The heat transfer coefficients at the upper and lower plates are respectively given by,

$$H_U = - \frac{L}{T_L'} \frac{dT_1'}{dz'}\bigg|_{z'=L} = - \frac{d\theta_1'}{dz}\bigg|_{z=1} = \frac{2P_1 E_c}{a_6 \mu^3 \alpha^3} [A_1\, \text{Sh}\, 2\alpha\mu$$

$$- B_1 \sin 2\alpha\mu + A_2\, \text{Ch}\, 2\alpha\mu + B_2 \cos 2\alpha\mu] - B_3 \tag{37}$$

$$H_L = \frac{L}{T'_L} \left. \frac{dT'_2}{dz'} \right|_{z'=-L} = \left. \frac{d\theta'_2}{dz} \right|_{z=-1} = B_6 - \frac{2P_1 E_c}{a_6 \mu^2 \lambda^2 \eta \alpha^3} [- A_4 \text{ Sh } 2\alpha$$

$$+ B_4 \sin 2\alpha + A_5 \text{ Ch } 2\alpha + B_5 \cos 2\alpha] \tag{38}$$

As $\alpha \to 0$, we get from Eqns. (35) to (38),

$$\theta'_1 = A + Bz - \frac{P_1 E_c}{6 (\lambda \mu^2 + 1)^2} [3 (\lambda \mu^2 - 1)^2 z^2 - 4 (\lambda^2 \mu^4 - 1) z^3$$

$$+ 2 (\lambda \mu^2 + 1)^2 z^4] \tag{39}$$

$$\theta'_2 = C + Dz - \frac{P_1 E_c}{6 \eta \lambda^2 \mu^2 (\lambda \mu^2 + 1)^2} [3 (\lambda \mu^2 - 1)^2 z^2$$

$$- 4 (\lambda^2 \mu^4 - 1) z^3 + 2 (\lambda \mu^2 + 1)^2 z^4] \tag{40}$$

where

$$A = C = \frac{P_1 E_c (\lambda^4 \mu^6 - 2\lambda^3 \mu^4 + 9\lambda^2 \mu^2 + 9\lambda^2 \mu^4 - 2\lambda \mu^2 + 1)}{6 (\eta + 1) \lambda^2 \mu^2 (\lambda \mu^2 + 1)^2},$$

$$B = \eta D = \frac{\eta P_1 E_c}{6 (\eta + 1) (\lambda \mu^2 + 1)^2} \left[\lambda^2 \mu^4 - 2\lambda \mu^2 + 9 \right.$$

$$\left. - \frac{1}{\eta \lambda^2 \mu^2} (9\lambda^2 \mu^4 - 2\lambda \mu^2 + 1) \right] \tag{41}$$

and

$$H_U = - \frac{P_1 E_c}{6 (\lambda \mu^2 + 1)^2} \left[\frac{\eta}{1 + \eta} \left[\lambda^2 \mu^4 - 2\lambda \mu^2 + 9 - \frac{1}{\eta \lambda^2 \mu^2} \right. \right.$$

$$\left. \left. (9\lambda^2 \mu^4 - 2\lambda \mu^2 + 1) \right] - 2\lambda^2 \mu^4 - 4\lambda \mu^2 - 26 \right] \tag{42}$$

$$H_L = \frac{P_1 E_c}{6 (\lambda \mu^2 + 1)^2} \left[\frac{\lambda^2 \mu^4 - 2\lambda \mu^2 + 9}{\eta + 1} - \frac{(9\lambda^2 \mu^4 - 2\lambda \mu^2 + 1)}{\eta (\eta + 1) \lambda^2 \mu^2} \right.$$

$$\left. + \frac{26\lambda^2 \mu^4 + 4\lambda \mu^2 + 2}{\eta \lambda^2 \mu^2} \right]. \tag{43}$$

Fig. 6 shows the temperature distribution for various values of λ, μ, η and α. It is concluded that the temperature at any point of the channel decreases with an increase in any of the parameters. The heat transfer coefficients at the upper and lower plates have been shown in Figs. 7 and 8 for above parameters respectively. It is concluded that the heat transfer coefficient at either plate decreases with an increase in any of the parameters.

Plates Maintained at Different Temperatures

It is assumed that the temperature of the upper plate T'_U is greater than the temperature of the lower plate T'_L. In terms of the non-dimensional quantities,

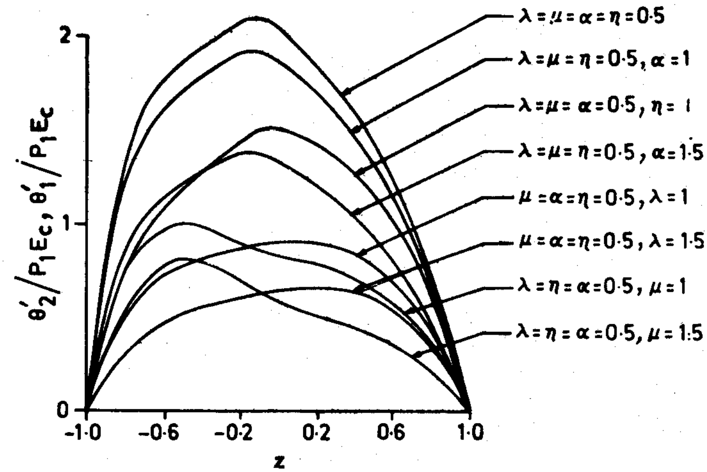

Figure 6. Temperature profiles.

$$\left.\begin{aligned}\theta'_1 &= \frac{T'_1 - T'_L}{T'_U - T'_L}, \quad \theta'_2 = \frac{T'_2 - T'_L}{T'_U - T'_L},\\ E_c \text{ (Eckert number for the upper fluid)} &= \left(\frac{P'_1 L^2}{2\rho_1 \nu_1}\right)^2 \Big/ C_p (T'_U - T'_L)\end{aligned}\right\} \tag{44}$$

We have to solve Eqn. (31) subject to the boundary conditions of Eqn. (32) except when $\theta'_1 = 1$ at $z = 1$.

Solving, we get,

$$\theta'_1 = A'_3 + B'_3 z - \frac{P_1 E_c}{a_6 \mu^4 \alpha^4} [A_1 \operatorname{Ch} 2\mu\alpha z + B_1 \cos 2\mu\alpha z + A_2 \operatorname{Sh} 2\mu\alpha z + B_2 \sin 2\mu\alpha z] \tag{45}$$

$$\theta'_2 = A'_6 + B'_6 z - \frac{P_1 E_c}{a_6 \mu^2 \lambda^2 \eta \alpha^4} [A_4 \operatorname{Ch} 2\alpha z + B_4 \cos 2\alpha z + A_5 \operatorname{Sh} 2\alpha z + B_5 \sin 2\alpha z] \tag{46}$$

where

$$\left.\begin{aligned}A'_3 &= A_3 + \frac{1}{1+\eta}, \quad B'_3 = B_3 + \frac{\eta}{1+\eta},\\ A'_6 &= A_6 + \frac{1}{1+\eta}, \quad B'_6 = B_6 + \frac{1}{1+\eta}.\end{aligned}\right\} \tag{47}$$

The heat transfer coefficients at the upper and lower plates are respectively given by

$$H_U = \frac{2P_1 E_c}{a_6 \mu^3 \alpha^3} [A_1 \operatorname{Sh} 2\mu\alpha - B_1 \sin 2\mu\alpha + A_2 \operatorname{Ch} 2\mu\alpha + B_2 \cos 2\mu\alpha] - B'_3 \tag{48}$$

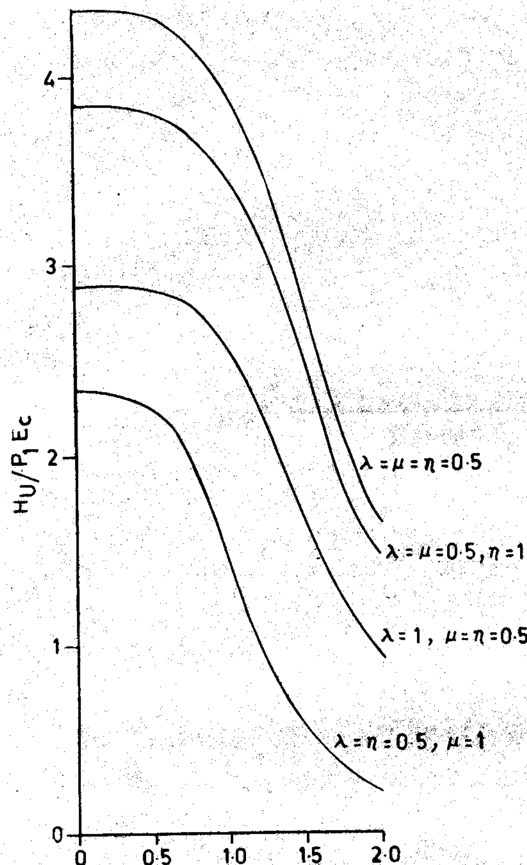

Figure 7. Heat transfer coefficient at the upper plate.

$$H_L = B_6' - \frac{2P_1E_c}{a_6\mu^2\lambda^2\eta\alpha^3}[-A_4 \text{ Sh } 2\alpha + B_4 \sin 2\alpha + A_5 \text{ Ch } 2\alpha + B_5 \cos 2\alpha] \tag{49}$$

As $\alpha \to 0$, we get from Eqns. (45) and (46),

$$\theta_1' = A' + B'z - \frac{P_1E_c}{6(\lambda\mu^2+1)^2}[3(\lambda\mu^2-1)^2 z^2 - 4(\lambda^2\mu^4-1)z^3 + 2(\lambda\mu^2+1)^2 z^4] \tag{50}$$

$$\theta_2' = C' + Dz' - \frac{P_1E_c}{6(\lambda\mu^2+1)^2\eta\lambda^2\mu^2}[3(\lambda\mu^2-1)^2 z^2 - 4(\lambda^2\mu^4-1)z^3 + 2(\lambda\mu^2+1)^2 z^4] \tag{51}$$

where

$$A' = C' = A + \frac{1}{1+\eta}, \quad B' = \eta D' = B + \frac{\eta}{1+\eta} \tag{52}$$

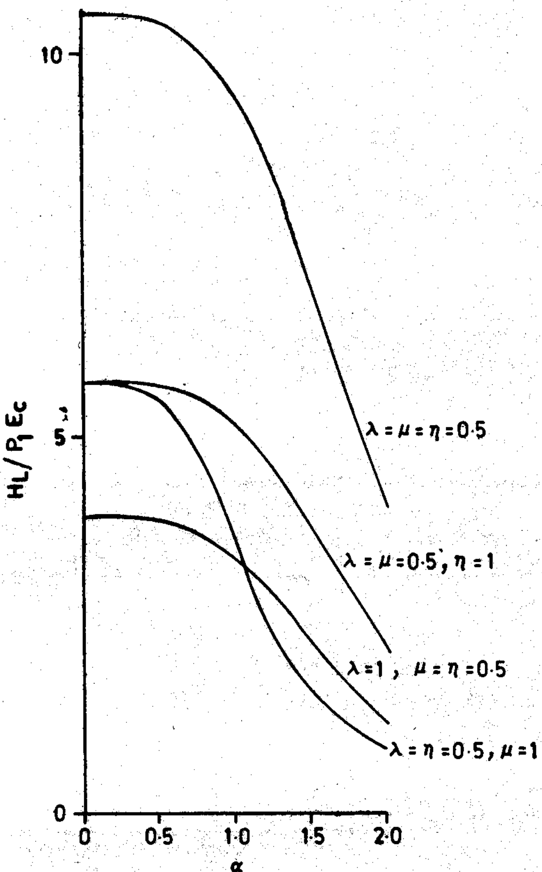

Figure 8. Heat transfer coefficient at the lower plate.

As $\alpha \to 0$, we get from Eqns. (48) and (49),

$$H_U = -\frac{\eta}{1+\eta} - \frac{P_1E_c}{6(\lambda\mu^2+1)^2}\left[\frac{\eta}{1+\eta}\left[\lambda^2\mu^4 - 2\lambda\mu^2 + 9 - \frac{1}{\eta\lambda^2\mu^2}(9\lambda^2\mu^4 - 2\lambda\mu^2 + 1)\right] - 2\lambda^2\mu^4 - 4\lambda\mu^2 - 26\right] \tag{53}$$

$$H_L = \frac{1}{1+\eta} + \frac{P_1E_c}{6(\lambda\mu^2+1)^2}\left[\frac{\lambda^2\mu^4 - 2\lambda\mu^2 + 9}{\eta+1} - \frac{(9\lambda^2\mu^4 - 2\lambda\mu^2 + 1)}{\eta(\eta+1)\lambda^2\mu^2} + \frac{26\lambda^2\mu^4 + 4\lambda\mu^2 + 2}{\eta\lambda^2\mu^2}\right] \tag{54}$$

It follows from Eqn. (48) that when $E_c = E_c^*$, where

$$E_c^* = \frac{B_3' a_6 \mu^3 \alpha^3}{2[A_1 \operatorname{Sh} 2\mu\alpha - B_1 \sin 2\mu\alpha + A_2 \operatorname{Ch} 2\mu\alpha + B_2 \cos 2\mu\alpha]\, P_1} \tag{55}$$

then there is no flow of heat either from plate to the fluid or from fluid to the plate.

Tables 1–6 shows the temperature distribution for various values of λ, μ, α, η, $P_1 = 0.72$ and $E_c = 0.02$. It is found that the conclusions of the temperature distribution of the previous case i.e., when the plate temperatures are equal, are also valid in this case. Tables 7–8 shows the calculated values of the coefficient of heat transfer

Table 1. Temperature distribution when $\lambda = \mu = \eta = 0.5$, $\alpha = 0$

z	—1.0	—0.8	—0.6	—0.4	—0.2	0.0
(θ_2'/P_1E_c)	0.0	10.598	20.296	29.732	39.133	48.317
z	0.0	0.2	0.4	0.6	0.8	1.0
(θ_1'/P_1E_c)	48.317	52.756	57.140	61.422	65.547	69.444

Table 2. Temperature distribution when $\lambda = \mu = \eta = 0.5$, $\alpha = 0.5$

z	—1.0	—0.8	—0.6	—0.4	—0.2	0.0
(θ_2'/P_1E_c)	0.0	10.589	20.284	29.719	39.119	48.304
z	0.0	0.2	0.4	0.6	0.8	1.0
(θ_1'/P_1E_c)	48.304	52.745	57.129	61.414	65.542	69.444

Table 3. Temperature distribution when $\lambda = \mu = \eta = 0.5$, $\alpha = 1$

z	—1.0	—0.8	—0.6	—0.4	—0.2	0.0
(θ_2'/P_1E_c)	0.0	10.474	20.131	29.550	38.937	48.128
z	0.0	0.2	0.4	0.6	0.8	1.0
(θ_1'/P_1E_c)	48.128	52.586	56.992	61.306	65.478	69.444

Table 4. Temperature distribution when $\lambda = \mu = \alpha = 0.5$, $\eta = 1$

z	—1.0	—0.8	—0.6	—0.4	—0.2	0.0
(θ_2'/P_1E_c)	0.0	7.710	14.973	22.105	29.220	36.228
z	0.0	0.2	0.4	0.6	0.8	1.0
(θ_1'/P_1E_c)	36.228	43.084	49.884	56.584	63.127	69.444

Table 5. Temperature distribution when $\lambda = \alpha = \eta = 0.5$, $\mu = 1$

z	−1.0	−0.8	−0.6	−0.4	−0.2	0.0
(θ_2'/P_1E_c)	0.0	10.002	19.500	28.760	37.943	47.117
z	0.0	0.2	0.4	0.6	0.8	1.0
(θ_1'/P_1E_c)	47.117	51.696	56.252	60.759	65.174	69.444

Table 6. Temperature distribution when $\alpha = \eta = \mu = 0.5$, $\lambda = 1.5$

z	−1.0	−0.8	−0.6	−0.4	−0.2	0.0
(θ_2'/P_1E_c)	0.0	9.597	19.006	28.332	37.639	46.941
z	0.0	0.2	0.4	0.6	0.8	1.0
(θ_1'/P_1E_c)	46.941	51.581	56.191	60.738	65.175	69.444

Table 7. Coefficient of heat transfer at the upper plate $(-H_U/P_1E_c)$

λ	μ	η \ α	0	0.5	1	1.5
0.5	0.5	0.5	18.788	18.816	19.194	20.278
0.5	0.5	1.0	30.867	30.892	31.226	32.184
0.5	1.0	0.5	20.796	20.888	21.714	22.578
1.0	0.5	0.5	20.257	20.282	20.604	21.424

Table 8. Coefficient of heat transfer at the lower plate (H_L/P_1E_c)

λ	μ	η \ α	0	0.5	1	1.5
0.5	0.5	0.5	56.811	56.745	55.857	53.304
0.5	0.5	1.0	40.485	40.449	39.960	38.558
0.5	1.0	0.5	52.037	51.834	49.987	48.011
1.0	0.5	0.5	50.247	50.212	49.752	48.587

Table 9. Critical Eckert number E_c^*

λ	μ	η	P_1 \ α	0.0	0.5	1.0	1.5
0.5	0.5	0.5	0.72	0.106	0.143	0.157	0.219
			1.00	0.076	0.101	0.111	0.156
			2.00	0.038	0.046	0.051	0.074
			4.00	0.019	0.019	0.022	0.033
0.5	0.5	1.0	0.72	0.180	0.222	0.244	0.337
			1.00	0.130	0.158	0.174	0.241
			2.00	0.065	0.077	0.084	0.118
			4.00	0.032	0.036	0.040	0.057
0.5	1.0	0.5	0.72	0.197	0.273	0.441	1.197
			1.00	0.142	0.194	0.315	0.859
			2.00	0.071	0.093	0.154	0.425
			4.00	0.035	0.043	0.073	0.207
1.0	0.5	0.5	0.72	0.160	0.153	0.172	0.254
			1.00	0.115	0.111	0.124	0.183
			2.00	0.058	0.056	0.063	0.092
			4.00	0.029	0.029	0.032	0.047
1.5	0.5	0.5	0.72	0.193	0.159	0.183	0.283
			1.00	0.139	0.116	0.133	0.204
			2.00	0.069	0.060	0.069	0.104
			4.00	0.035	0.032	0.037	0.054

at the upper and lower plates respectively. As in the previous case the heat transfer coefficient at either plate is found to decrease with an increase in any of the parameters. Table 9 shows the Eckert number E_c^* for various values of λ, μ, α, η and P_1. It is observed that this number generally increases with increase in α, λ and η whereas it decreases with increase in P_1.

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