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Stability of Dusty Flow Between Two Rotating Coaxial Cylinders

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Abstract. The criterion for stability of dusty fluid flow between two coaxially rotating cylinders is derived by an examination of the stability of the basic flow to arbitrary perturbations. The perturbation equations are explored under the assumption that the relaxation time $\tau \left(= \frac{m}{k^*}\right)$ of the dust particles is small. It is observed that the flow is always stable if Rayleigh's criterion $\mu > \eta^2$ is satisfied. The principle of exchange of stabilities is discussed and the characteristic value problem is solved under the narrow-gap approximations.

1. Introduction

Saffman[1] gave a formulation of the problem of the linearized stability of a plane parallel flow of a dusty gas; in which the dust is represented macroscopically in terms of a number density of very small particles. Following Saffman, some approximate results were obtained by Michael[2] for the problem of the stability of plane Poiseuille flow of a dusty gas. Michael and Norey[3] considered the stability of laminar flow of dusty gas between two rotating cylinders which start to rotate impulsively from rest. Assuming that the relaxation time τ is small, they derived perturbation equations and obtained solutions by considering the ratio of the time scales, on which the gas velocity and the mass concentration of the dust change, to be both large and small. The stability of Coutte-flow was considered by Chandrasekhar[4] in which the boundaries are two coaxial cylinders and the basic flow is circular. He established the criterion for stability by examining the stability of basic flow for arbitrary perturbation. Analysing the disturbances into normal modes he obtained the solution of the perturbation equations which are of the form $U = e^{pt}U(r) \cos kz$ where p is a constant (which can be complex) and k is the wave number of disturbance in z-direction.

In this paper the stability of dusty fluid flow between two coaxial cylinder rotating with different angular velocities is investigated using the aforesaid method of Chandrasekhar. The perturbation equations are derived by assuming that the time relaxation of dust particles is small. The principle of exchange of stabilities is discussed and the characteristic value problem is solved under the narrow gap approximations.

It is interesting to note that Rayleigh's criterion $\mu > \eta^2$ is a sufficient condition for stability of dusty Coutte-flows also.

2. Basic Equations

The hydrodynamical equations governing viscous incompressible fluid in cylindrical polar co-ordinates (r, θ, z) are given in components form

$$\frac{\partial u_r}{\partial t} + (\vec{u} \, . \, \text{grad}) \, u_r - \frac{u_\theta^2}{r} = - \frac{\partial}{\partial r} (\rho^*/\rho) + \nu \left(\nabla^2 u_r - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right) + \frac{k^* N}{\rho} (v_r - u_r) \tag{1}$$

$$\frac{\partial u_\theta}{\partial t} + (\vec{u} \, . \, \text{grad}) \, u_\theta + \frac{u_\theta u_r}{r} = - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{p^*}{\rho} \right) + \nu \left(\nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) + \frac{k^* N}{\rho} (v_\theta - u_\theta) \tag{2}$$

$$\frac{\partial u_z}{\partial t} + (\vec{u} \, . \, \text{grad}) \, u_z = - \frac{\partial}{\partial z} \left(\frac{p^*}{\rho} \right) + \nabla^2 u_z + \frac{k^* N}{\rho} (v_z - u_z) \tag{3}$$

and the other set of similar equations for the dust particles are given by

$$\frac{\partial v_r}{\partial t} + (\vec{v} \, . \, \text{grad}) \, v_r - \frac{v_\theta^2}{r} = \frac{k^*}{m} (u_r - v_r) \tag{4}$$

$$\frac{\partial v_\theta}{\partial t} + (\vec{v} \, . \, \text{grad}) \, v_\theta + \frac{v_\theta v_r}{r} = \frac{k^*}{m} (u_\theta - v_\theta) \tag{5}$$

$$\frac{\partial v_z}{\partial t} + (\vec{v} \, . \, \text{grad}) \, v_z = \frac{k^*}{m} (u_z - v_z) \tag{6}$$

where

$$(\vec{u} \, . \, \text{grad}) = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

and ∇^2 stands for

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

We also have the equation of continuity

$$\frac{\partial u_r}{\partial t} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{7}$$

In these equations t is the time, ρ the density, ν the kinematic viscosity, p^* the pressure, k^* the stokes resistance co-efficient, m the mass of dust particle, N the number density of dust particles and u_r, u_θ, u_z are the components of the velocity $\vec{u}$ of clean fluid, v_r, v_θ, v_z are the components of the velocity $\vec{v}$ of dust particles.

These equation allow a stationary solution of the form

$$\left.\begin{aligned} u_r &= u_z = 0 \text{ and } u_\theta = U(r) \\ v_r &= v_z = 0 \text{ and } v_\theta = V(r) \end{aligned}\right\} \tag{8}$$

For these solutions Eqns. (1) to (6) give

$$U = V = r\Omega = A_r + \frac{B}{r} \tag{9}$$

where Ω is the angular velocity of rotation about the axis and

$$A = \frac{\Omega^2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \; ; \; B = -\frac{(\Omega_2 - \Omega_1)\, R_1^2 R_2^2}{R_2^2 - R_1^2}$$

Ω_1, R_1 and Ω_2, R_2 are the angular velocity and radii of the inner and outer cylinders respectively.

3. The Perturbation Equations

We shall investigate here the stability of the flow described by the Eqn. (9), let the perturbed state be characterised by

$$u_r,\ U + u_\theta,\ u_z,\ \frac{\delta p}{\rho} = \bar{\omega} \tag{10}$$

$$v_r,\ U + v_\theta,\ v_z.$$

Again let us assume that the t and z dependence of the perturbations are given by

$$\left.\begin{aligned} u_r &= e^{pt}\, u_r(r) \cos kz\,, & u_\theta &= e^{pt}\, u_\theta(r) \cos kz \\ u_z &= e^{pt}\, u_z(r) \sin kz\,, & \bar{\omega} &= e^{pt}\, \bar{\omega}(r) \cos kz \\ v_r &= e^{pt}\, v_r(r) \cos kz\,, & v_\theta &= e^{pt}\, v_\theta(r) \cos kz \\ v_z &= e^{pt}\, v_z(r) \sin kz\,, \end{aligned}\right\} \tag{11}$$

where k is the wave number of the disturbance in the axial direction and p is constant, which can be complex. For solution of the form Eqn. (11) the linearized perturbation equations are

$$\nu\left(DD^* - k^2 - \frac{p}{\nu}\right) u_r + \frac{2U}{r}\, U_\theta = \frac{\partial \bar{\omega}}{\partial r} - \frac{K^*N}{\rho}\,(v_r - u_r) \tag{12}$$

$$\nu\left(DD^* - k^2 - \frac{p}{\nu}\right) u_\theta - (D^*U)\, u_r + \frac{k^*N}{\rho}\,(v_\theta - u_\theta) = 0 \tag{13}$$

$$\nu\left(DD^* - k^2 - \frac{p}{\nu}\right) u_z + \frac{k^*N}{\rho}(v_z - u_z) = -k\bar{\omega} \tag{14}$$

$$pv_r - \frac{2U}{r} v_\theta = \frac{k^*}{m}(u_r - v_r) \tag{15}$$

$$pv_\theta + \left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) v_r = \frac{k^*}{m}(u_\theta - v_\theta) \tag{16}$$

$$pv_z = \frac{k^*}{m}(u_z - v_z) \tag{17}$$

$$D^* u_r = -ku_z \tag{18}$$

Eliminating u_z, v_z and $\bar{\omega}$ from Eqns. (12), (14), (17) and (18), we get after some rearrangements that

$$\frac{\nu}{k^2}\left[\left(DD^* - k^2 - \frac{p}{\nu} - \frac{s}{\nu}\right)(DD^* - k^2)\right] u_r$$

$$= \frac{2U}{r} u_\theta + \frac{k^*N}{\rho}(v_r - u_r) + s\, u_r \tag{19}$$

where

$$s = \frac{p_m\, k^*N}{(k^* + p_m)}$$

This equation must be considered together with

$$\left(DD^* - k^2 - \frac{p}{\nu}\right) u_\theta = (D^*U)\, u_r - \frac{k^*N}{\rho}(v_\theta - u_\theta) \tag{20}$$

where

$$D = \frac{d}{dr},\; D^* = \left(\frac{d}{dr} + \frac{1}{r}\right).$$

Taking $\frac{m}{k^*} = \tau$, the time relaxation parameter, to be small, we can ignore the terms involving the product of τ. Under this approximation Eqns. (15) to (17) yield

$$u_r = v_r,\quad u_\theta = v_\theta,\quad u_z = v_z.$$

Thus the Eqns. (19) and (20) reduce to

$$\left[\frac{\nu}{k^2}\left(DD^* - k^2 - \frac{p}{\nu} - \frac{s}{\nu}\right)(DD^* - k^2) - s\right] u_r = \frac{2U}{r} u_\theta \tag{21}$$

and

$$\left(DD^* - k^2 - \frac{p}{\nu}\right) u_\theta = (D^*U)\, u_r \tag{22}$$

Measuring r in the units of R_2 the outer cylinder's radius and writing $k^2 = \frac{a^2}{R_2^2}$ and

$\sigma = \frac{pR_2^2}{\nu}$ and replacing $\frac{2AR_2^2}{\nu} u_\theta$ by u_θ and $\frac{2aR_1^2}{\nu} u_r$ by u_r the Eqns. (21) and (22) with the help of Eqn. (9) finally reduce to

$$\left[(DD^* - a^2 - \sigma - \lambda)(DD^* - a^2) - a^2\lambda\right] u_r = - Ta^2 \left(\frac{1}{r} - \bar{k}\right) u_\theta \tag{23}$$

and

$$(DD^* - a^2 - \sigma) u_\theta = u_r \tag{24}$$

where

$$T = 4\Omega_1^2 R_1^4 (1 - \mu)\left(1 - \frac{\mu}{\eta^2}\right)$$

$$\bar{k} = - \frac{AR_2^2}{B} = - \frac{1 - \frac{\mu}{\eta^2}}{1 - \mu}, \quad \frac{SR_2^2}{\nu} = \lambda$$

$$\mu = \frac{\Omega_2}{\Omega_1}, \quad \eta = \frac{R_2}{R_1}.$$

The solutions of Eqns. (23) and (24) must be investigated which satisfy the boundary conditions suitable for no slip on the cylinderical walls at $r = 1$ and $r = \eta$. These conditions are that all the components of the velocity vanish on the walls. Thus $u_r = u_\theta = Du_r = 0$ for $r = 1$ and $r = \eta$.

4. Stability of the Flow for $\mu > \eta^2$

Now we shall show that when Rayleigh's criterion $\mu > \eta^2$ is satisfied, the flow is naturally stable. For this we multiply Eqn. (23) by ru_r^* (the complex conjugate of u_r) and integrate over the range of r.

Thus we have

$$\int_\eta^1 [(DD^* - a^2 - \sigma - \lambda)(DD^* - a^2) - a^2\lambda] u_r d_r$$

$$= - T'a^2 \int_\eta^1 ru_r^* \phi(r) u_\theta dr \tag{25}$$

where

$$T = 4\Omega_1 R_1^4 \frac{\left(1 - \frac{\mu}{\eta^2}\right)}{(1 - \eta^2)^2}$$

and

$$\phi(r) = (1 - \mu)\left(\frac{1}{r^2} - \bar{k}\right)$$

since u_r and its derivatives vanish at $r = 1$ and $r = \eta$, the integrals on the left hand side of Eqn. (25) can be transformed to positive definite forms by Chandrasekhar[4] (ref. Eqns. 172 and 173 p. 296). Further substituting for u_r from Eqn. (24) in the integral on the right hand side of Eqn. (25) and making the use of Chandrasekhar[4] (ref. pp. 296–97), finally we get

$$(\sigma + \lambda) I_1 + I_2 - a^2\lambda I_5 = T' a^2 [(a^2 + \sigma^*) I_3 + I_4] \tag{26}$$

where

$$I_1 = \int_\eta^1 \left\{ r \left| \frac{du_r}{dr} \right|^2 + \left(\frac{1}{r^2} + a^2 r \right) \left| u_r \right|^2 \right\} dr$$

$$I_2 = \int_\eta^1 | (DD^* - a^2) u_r |^2 dr$$

$$I_3 = \int_\eta^1 r\phi(r) | u_\theta |^2 dr$$

$$I_4 = \int_\eta^1 \phi(r) \left\{ r \left| \frac{du_\theta}{dr} \right|^2 + \frac{| u_\theta |^2}{r} \right\} dr - 2(1 - \mu) \int_\eta^1 \frac{u_\theta}{r^2} \frac{du_\theta^*}{dr} dr$$

$$I_5 = \int_\eta^1 r | u_r |^2 dr.$$

Following the arguments of Chandrasekhar[4] (ref. pp. 279–298) and equating the real parts of Eqn. (26), we have

$$\frac{R_2^2}{\nu} Re\,(p) \left[I_1 - T' a^2 I_3 + \frac{Ak^*}{m} I^* \right] + \frac{Ak^*}{m} I^* + \text{Positive terms} = 0$$

where

$$I^* = \int_\eta^1 \left\{ r \left| \frac{du_r}{dr} \right|^2 + \frac{1}{r} \left| u_r \right|^2 \right\} dr$$

Since $\mu > \eta^2$, $T' < 0$, we have $Re(p) > 0$ and the flow is therefore stable.

5. The Solution for the Case of a Narrow Gap when the Marginal State is Stationary

If the gap $(R_2 - R_1)$ between two cylinders is small in comparison with the mean radius $\frac{1}{2}(R_1 + R_2)$, we need not distinguish D and D^* in Eqns. (21) and (22). We can also replace $\Omega = \left(A + \frac{B}{r^2} \right)$ by $\Omega_1 \left[1 - (1 - \mu) \frac{r - R_1}{R_2 - R_1} \right]$.

By writing $k = \frac{a}{d}$, $\sigma = \frac{pd^2}{\nu}$, $\frac{sd^2}{\nu} = \lambda$ and using the transformation $u_r \to \frac{2\Omega_1 a^2 d^2}{\nu} u_r$, we find that the Eqns. (21) and (22) reduce to

$$[(D^2 - a^2 - \sigma - \lambda)(D^2 - a^2) - a^2\lambda]\, u_r = (1 + \alpha\zeta)\, u_\theta \tag{27}$$

and

$$(D^2 - a^2 - \sigma)\, u_\theta = -\, Ta^2 u_r \tag{28}$$

where

$$T = -\frac{4A\Omega_1}{\nu^2} d^2,\ d = -(1 - \mu)$$

$$\zeta = \frac{r - R_1}{R_2 - R_1} = \frac{r - R_1}{d}.$$

Equations (27) and (28) must be considered together with the boundary conditions.

$$u_r = Du_r = u_\theta = 0 \text{ for } \zeta = 1 \text{ and } \eta.$$

The case $Re(p) = 0$ corresponds to overstability. In the case of clean fluid, Chandrasekhar[4] has pointed out that experiments on onset of stability have failed to reveal any suggestions of the overstability. Hence this case is not discussed. For $\sigma = 0$, we have $\lambda = 0$ and Eqns. (27) and (28) reduced to chandrasekhar[4] (ref. Eqns. (201) and (202), p. 300). Therefore in this case his conclusions (pp. 330–315) for clean fluid hold equally good for dusty fluid.

6. The Principle of the Exchange of Stabilities

Experiments on the stability of coutte flow of clean fluid suggest that instability sets in as secondary stationary flow. To examine the validity of the principle of exchange of stabilities, we translate the origin of coordinates midway between the two cylinders and replace u_r by $\frac{u_r(1+\mu)}{2}$. We see that the Eqns. (27) and (28) reduced to

$$[(D^2 - a^2 - \sigma - \lambda)(D^2 - a^2) - a^2\lambda]\, u_r = (1 + \epsilon \cdot x)\, u_\theta \tag{29}$$

and

$$(D^2 - a^2 - \sigma)\, u_\theta = -\, \bar{T}a^2 u_r \tag{30}$$

where

$$\bar{T} = \tfrac{1}{2}(1 + \mu)\, T,\ \epsilon = -2\frac{(1 - \mu)}{1 + \mu}$$

consider at first the case $\mu > 0$

As in the case of clean fluid, we ignore the first order terms in ϵ on the right hand side of Eqn. (29) and we get

$$[(D^2 - a^2 - \sigma - \lambda)(D^2 - a^2) - a^2\lambda]\, u_r = u_\theta \tag{31}$$

and

$$(D^2 - a^2 - \sigma)\, u_\theta = -\, \bar{T}a^2 u_r \tag{32}$$

together with the boundary conditions

$$u_r = u_\theta = Du_r = 0 \text{ for } x = \pm \tfrac{1}{2}$$

we multiply Eqn. (31) by u_r (the conjugate of u_r) and integrate over the range of x. Integrating by parts, we obtain

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} u_\theta u_r^* \, dx = \int_{-\frac{1}{2}}^{+\frac{1}{2}} |(D^2 - a^2) u_r|^2 dx + (\sigma + \lambda) \int_{-\frac{1}{2}}^{+\frac{1}{2}} \{ |Du_r|^2 + a^2 |u_r|^2 \} dx$$

$$- a^2\lambda \int_{-\frac{1}{2}}^{+\frac{1}{2}} |u_r|^2 dx \tag{33}$$

From Eqn. (32), we find that

$$- \bar{T}a^2 \int_{-\frac{1}{2}}^{+\frac{1}{2}} u_r^* u_\theta \, dx = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \{ |Du_\theta|^2 + (a^2 + \sigma^*) |u_\theta|^2 \} dx \tag{34}$$

combining the Eqns. (33) and (34) together, we find a relation independant of λ which is exactly the same as the equation of Chandrasekhar[4] (ref. Eqn. 285 p. 316). From which we conclude that

$$I_m(\sigma) = 0 \quad \text{when } \bar{T} > 0$$

and therefore the principle of stabilities is valid.

For negative values of μ, the replacement of Eqns. (29) and (30) by (31) and (32) leads to considerable errors. As in the case of clean fluid the possibility of over-stability cannot be ruled out by the above discussion. Hence we solve the characteristic value problem represented by Eqns. (27) and (28) with the boundary conditions.

$$U_r = u_\theta = Du_r, = 0 \text{ for } \zeta = 1 \text{ and } \eta.$$

7. The Solution of the Characteristic Value Problem for the Case $\sigma \neq 0$

It can be easily verified that the characteristic value problem presented by Eqns. (27) and (28) is not self-adjoint in the usual sense. To solve this problem we follow the orthogonal developement of Chandrasekhar[4].

Since u_θ is required to vanish at $\zeta = 0$ and 1, we expand it in a sine series of the form

$$u_\theta = \sum_{m=1}^{\infty} Cm \cdot \sin m\pi\zeta \tag{35}$$

Having taken u_θ in this manner, we solve the equation

$$[(D^2 - a^2 - \sigma - \lambda)(D^2 - a^2) - a^2\lambda] u_r = (1 + \alpha\zeta) \sum_{m=1}^{\infty} Cm \cdot \sin m\pi\zeta \tag{36}$$

obtained by inserting Eqn. (35) in Eqn. (27) and arranging that the solution satisfies the four remaining boundary conditions on u_r. With u_r determined in this way and u_θ given by Eqn. (35), Eqn. (28) will lead to a secular equation in T

$$\Big\| \frac{mn\pi^2}{(n^2\pi^2 + m_1^2)} \frac{1}{(\cosh m_1 - \cosh m_2)} [1 + (-1)^{m+1} \cosh m_1]$$
$$\times \Big[Q((-1)^{m+1} + \cosh m_2) - \frac{L}{p} (m_1 \cosh m_2 - m_2 \cosh m_1)$$
$$- \frac{M}{p} \left(\sinh m_2 - \frac{m}{m_1} \sinh m_1 \right) + \frac{\sinh m_1}{m_1} \Big] - \frac{mn\pi^2}{(n^2\pi^2 + m_2^2)}$$
$$\times \frac{1}{(\cosh m_1 - \cosh m_2)} \Big[1 + (-1)^{m+1} \cosh m_2 \Big] \Big[Q\,((-1)^{m+1}$$
$$+ \cosh m_1) - \frac{L}{p} (m_1 \cosh m_2 - m_2 \cos m_1)$$
$$- \frac{M}{p} \left(\sinh m_2 - \frac{m_2}{m_1} \sinh m_1 \right) + \frac{\sinh m_1}{m_1} \Big]$$
$$- \frac{mn\pi^2}{(n^2\pi^2 + m_1^2)} (-1)^{n+1} \sinh m_1 \left[1 + \frac{m_2 m_1 L + m_2 M}{p} \right]$$
$$+ \frac{mn}{(n^2\pi^2 + m_2^2)} (-1)^{n+1} \sinh m_2 \left[\frac{m_1 L + M}{p} \right]$$
$$+ X_{nm} + \tfrac{1}{2}\delta_{nm} - \tfrac{1}{2} (n^2\pi^2 + a^2 + \sigma) \{(m^2\pi^2 + a^2)^2 + (\sigma + \lambda)$$
$$\times (m^2\pi^2 + a^2) - a^2\lambda\} \frac{\delta_{nm}}{a^2 T} \Big\| = 0 \qquad (37)$$

where

$$X_{nm} = \begin{cases} 0 \text{ if } m + n \text{ is an even, } m \neq n \\ \tfrac{1}{4} \text{ if } m = n \\ \dfrac{4mn}{n^2 - m^2} \left[\dfrac{2\left(m^2\pi^2 + a^2 + \dfrac{\sigma + \lambda}{2} \right)}{(m^2\pi^2 + a^2)^2 + (\sigma + \lambda)(m^2\pi^2 + a^2) - a^2\lambda} - \dfrac{1}{\pi^2(n^2 - m^2)} \right] \end{cases}$$

$$m_1, m_2 = \left[\frac{(2a^2 + \sigma + \lambda)}{2} \pm \frac{(\sigma + \lambda)}{2} \left(1 + \frac{4a^2}{\sigma + \lambda} - \frac{a^2}{(\lambda + \sigma)^2} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}$$

$$L = \frac{4\alpha(-1)^{m+1} \left(m^2 + a^2 + \dfrac{\sigma + \lambda}{2} \right)}{(m^2\pi^2 + a^2)^2 + (\sigma + \lambda)(m^2\pi^2 + a^2) - a^2\lambda}$$
$$\times (m_1 \sinh m_1 \cosh m_2 - m_2 \cosh m_1 \sinh m_2)$$
$$+ \frac{4\alpha \left(m^2\pi^2 + a^2 + \dfrac{\sigma + \lambda}{2} \right)}{(m^2\pi^2 + a^2)^2 + (\sigma + \lambda)(m^2\pi^2 + a^2) - a^2\lambda} \times$$

$$\times (m_1 \sinh m_1 \cosh m_2 - m_2 \cosh m_1 \sinh m_2)$$
$$- (-1)^{m+1} (1 + \alpha) (\cosh m_1 - \cosh m_2)$$

$$M = (\cosh m_1 \cosh m_2 - 1) m_1 - m_2 \sinh m_1 \sinh m_2$$

$$P = 2(1 - \cosh m_1 \cosh m_2) m_1 m_2 + (m_1^2 + m_2^2) \sinh m_1 \sinh m_2$$

$$Q = \frac{4\alpha \left(m^2\pi^2 + a^2 + \frac{\sigma + \lambda}{2} \right)}{(m^2\pi^2 + a^2)^2 + (\sigma + \lambda)(m^2\pi^2 + a^2) - a^2\lambda}$$

To draw any conclusion regarding the overstability one has to examine the roots of Eqn. (37). For given 'a' if there exists a real σ for which the characteristic root of Eqn. (37) is real, then overstability is possible. A first approximation to the solution of Eqn. (37) is obtained by setting the (1, 1) element of the matrix equal to zero.

This determines T. Thus

$$\tfrac{1}{2}(\pi^2 + a^2\sigma)\{(\pi^2 + a^2) + (\sigma + \lambda)(\pi^2 + a^2) - a^2\lambda\} \frac{1}{a^2T}$$
$$= \frac{\pi^2}{(\pi^2 + m_1^2)} \frac{(1 + \cosh m_1)}{\cosh m_1 - \cosh m_2} \Big[Q'(1 + \cosh m_2)$$
$$- \frac{L'}{P}(m_1 \cosh m_2 - m_2 \cosh m_1) - \frac{M}{P}$$
$$\times \left(\sinh m_2 - \frac{m_2}{m_1} \sinh m_1 \right) + \frac{\sinh m_1}{m_1} \Big]$$
$$- \frac{\pi^2}{(\pi^2 + m_2^2)} \frac{(1 + \cosh m_2)}{\cosh m_1 - \cosh m_2} \Big[Q'(1 + \cosh m_1)$$
$$- \frac{L'}{P}(m_1 \cosh m_2 - m_2 \cosh m_1) - \frac{M}{P}$$
$$\times \left(\sinh m_2 - \frac{m_2}{m_1} \sinh m_1 \right) + \frac{\sinh m_1}{m_1} \Big]$$
$$- \frac{\pi^2}{(\pi^2 + m_1^2)} \sinh m_1 \left(1 + \frac{(m_1 m_2 L' + m_2 M)}{P} + \frac{\pi^2}{(\pi^2 + m_2^2)} \sinh m_2 \right.$$
$$\times \frac{(m_1 L' + M)}{P} + \frac{\alpha}{4} + \tfrac{1}{2} \tag{38}$$

$$L' = \frac{4\alpha \left(\pi^2 + a^2 + \frac{\sigma + \lambda}{2} \right)(m_1 \sinh m_1 - m_2 \sinh m_2)}{(\pi^2 + a^2)^2 + (\sigma + \lambda)(\pi^2 + a^2) - a^2\lambda}$$
$$- (1 + \alpha)(\cosh m_1 - \cosh m_2)$$
$$+ \frac{4\alpha \left(\pi^2 + a^2 + \frac{\sigma + \lambda}{2} \right)}{(\pi^2 + a^2)^2 + (\sigma + \lambda)(\pi^2 + a^2) - a^2\lambda}$$
$$\times (m_1 \sinh m_1 \cosh m_2 - m_2 \cosh m_1 \sinh m_2).$$

$$Q' = \frac{4\alpha\left(\pi^2 + a^2 + \dfrac{\sigma + \lambda}{2}\right)}{(\pi^2 + a^2)^2 + (\sigma + \lambda)(\pi^2 + a^2) - a^2\lambda}$$

8. Discussion, Conclusion and Applications

The perturbation Equations (12) to (18) are not easily solvable as they stand. Under small gap approximations these are considerably simplified and some conclusions can be drawn. The only general result without any approximation is contained in section 4. This states that in the case of clean fluid the flow is stable when Rayleigh's criterion $\mu\left(=\frac{\Omega_2}{\Omega_1}\right) > \eta^2$ is satisfied. In otherwords, for stability, the outer cylinder must rotate with an angular velocity greater than η^2-times that of the inner-cylinder and in the same sense. Nevertheless, it appears to be only one which can be established on general analytical grounds. In particular, it does not seem that we can deduce the general validity of the principle of exchange of stabilities for this problm. No general conclusion can be drawn, when $\mu < 0$. However, from section 6, we conclude that the principle of exchange of stabilities is valid when $\mu > 0$. But no definite conclusion can be drawn from this section for $\mu < 0$. To get a definite conclusion which is valid for all positive and negative values of μ one has to solve the secular equation (37) and thus determines a critical value of T at which instability sets. The method of solution is indicated by Chandrasekhar[4] (ref. pp. 303–307).

In recent years, workers in the field of fluid dynamics have been paying attention to the study of dusty viscous fluid flows. The contamination of air in cities by dust particles has necessiated the study of the flow of dusty gases. The study of fluid or gas having uniform distribution of solid spherical particles is of interest in a wide range of technical importance. These areas include fluidization (flow through packed beds) environmental pollution, in the process by which drops of rain are formed by the coalscenes of small droplets which might be considered as solid particles for the purpose of examining their movement prior to coalscene, combustion and more recently, flow of blood. It is due to these reasons, a number of studies of flow of a fluid embeded with dust particles have appeared in the literature.

Engineering wise, such stability problems or similar ones may occur in liquid metal bearings for machines which are physically within closed loops using liquid-metal process fluids. Because of the low viscosity and high speed often associated with such machines, Taylor numbers are high. Since liquid metals are good conductors, magnetohydrodynamic effects could be significant.

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