Def Sci J, Vol 31, No 1, January 1981, pp 53-61

Couette Flow of a Viscous Electrically Conducting Fluid in a Porous Annulus

S. C. GUPTA & P. C. JAIN

Department of Mathematics, Vardhaman College, Bijnor-246701

Received 24 April 1978 ; revised 11 October 1979

Abstract. In the present paper the author has studied the Couette flow of a viscous incompressible fluid under a transverse magnetic field in an annular space bounded by two porous coaxial cylinders which are moving with arbitrary time dependent velocities parallel to the axis.

Introduction

Flow through porous media is encountered in a broad range of scientific and engineering activity which includes such diverse fields as solid mechanics, filteration, petroleum engineering and transpiration cooling. In recent years, the problems of fluid flow in channels with mass transfer at the boundaries have attracted the attention of mathematicians and engineers because of possible nuclear aero-dynamics and rocket engine applications. Singh[1] has considered the problems of impulsive motion of viscous fluid contained between the two porous concentric cylinders in the hydromagnetic case both in presence of radial and axial magnetic field. Muhuri[2] has considered the magneto-hydrodynamic flow between the parallel porous plates when one wall has given an impulsive or uniformly accelerated motion with suction at both the plates. Recently, Mathur[3] has considered the unsteady flow of a viscous incompressible fluid between two porous plates under a transverse magnetic field when the rate of injection of the fluid at the lower plate is equal to the rate of suction at the upper plate when the upper plate is given (i) uniform and (ii) impulsive motion.

In the present paper, the authors have studied the Couette flow of a viscous incompressible fluid under a transverse magnetic field in an annular space bounded by two porous coaxial cylinders which are moving about its axis with arbitrary time dependent velocities. This type of problems have great importance in the field of plasma physics as well as in astrophysics. The exact solution of the problem has been obtained with the help of finite Hankel transform which has an advantage over that of Laplace transform in the sense that laborious calculations and complications do not arise.

In fact, the general series solution of the problem for long straight channels with uniform· concentric cross section is given for all values of cross-flow Reynolds number and Hartmann number. The assumptions made for solving the problem are as follows :

(i) Initially, fluid and the cylinders should be at rest.

(ii) Fluid starts moving only if the cylinders are given time varying velocities about z'-axis.

(iii) The rate of injection of the fluid at one cylinder is equal to the rate of suction of the fluid at the other.

(iv) The two cylinders should be of uniform porosity.

Nomenclature

H_0 Strength of the imposed magnetic field

m $\mu_e H_0 a(\sigma_e/\rho\nu)^{1/2}$ Hartmann number

μ_e magnetic permeability

σ_e electric conductivity

ν kinematic viscosity

ρ density

u_z axial velocity

u_r radial velocity

v_a radial velocity at the inner cylinder

v_b radial velocity at the outer cylinder

R $\frac{av_a}{\nu}$ cross-flow Reynolds number

a radius of inner cylinder

b radius of outer cylinder

σ b/a.

Formulation of the Problem

Let a and b be the radii of the inner and outer cylinders respectively. Taking centre of the cylinders as origin and z'-axis along the axis of the cylinders, a frame of cylindrical polar system of coordinates is taken for reference (r', θ, z'). Let u_r, $U_{\bar{\theta}}$, u_z be the components of the velocity in the direction of r', θ' and z' respectively. For the present problem it is assumed that the pressure gradient is zero and flow is produced by the motion of the cylinders which are moving in z'-direction. Since we have considered the motion of the cylinders in z'-direction only, there is no displacement of the fluid in the direction of r' and θ' due to the motion of the cylinders. Hence assuming the azimuthal component of the velocity to be zero, the equations of motion for the problem neglecting the electromagnetic induced effect, Pai[4], are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - m^2 u, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial r} = \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} \tag{2}$$

and

$$\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \tag{3}$$

where

$$r = r'/a,\ z = z'/a,\ t = t'\nu/a,\ u = au_z/\nu \tag{4}$$

and $\quad v = au_r/\nu$

are dimensionless variables.

Since the flow is produced by the motion of the cylinders, let us assume that inner and outer cylinders start to move from rest with velocities $\phi_1(t)$ and $\phi_2(t)$ where $\phi_1(t)$ and $\phi_2(t)$ are functions of time. Therefore, the initial and boundary conditions for the problem are

Initial condition :

$$u = 0,\ v = 0 \text{ for } t \leqslant 0 \tag{5}$$

Boundary conditions :

$$\left.\begin{aligned} u &= \phi_1(t),\ v = \frac{av_a}{\nu} \text{ at } r = 1 \\ u &= \phi_2(t),\ v = \frac{av_b}{\nu} \text{ at } r = \sigma \end{aligned}\right\} \text{ for } t > 0 \tag{6}$$

Solution

In the present investigation, it is assumed that the rate of injection of the fluid at one wall is equal to rate of suction at the other wall and these rates are independent of time t and the axial position z. The condition that the rate of injection of the fluid at one cylinder is equal to rate of suction of the fluid at the other cylinder gives

$$av_a = bv_b \tag{7}$$

In view of uniform porosity, the axial component u is independent of z. Therefore Eqn. (3) reduces to

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0, \tag{8}$$

the solution of which under second condition of Eqn. (6) is

$$v = \frac{av_a}{\nu r} = \frac{bv_b}{\nu r} \tag{9}$$

Substituting the value of ν from Eqn. (9) in Eqn. (1), we get

$$\frac{\partial u}{\partial t} + \frac{R}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - m^2 v \tag{10}$$

Using the substitution

$$u(r, t) = r^{R/2}\,\omega(r, t) \tag{11}$$

it can be shown that the Eqn. (10) in the new variable ω is

$$\frac{\partial \omega}{\partial t} = \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} - \frac{R^2\omega}{4r^2} \tag{12}$$

and the initial and boundary conditions are

$$t \leqslant 0,\ \omega = 0 \text{ for } 1 \leqslant r \leqslant \sigma \tag{13}$$

$$\left.\begin{aligned} t > 0,\ \omega &= \phi_1(t) \text{ at } r = 1 \\ \omega &= \sigma^{-R/2}\,\phi_2(t) \text{ at } r = \sigma \end{aligned}\right\} \tag{14}$$

Let $\bar{\omega}$ denote the finite Hankel of ω defined by

$$\bar{\omega}(p_i, t) = \int_1^\sigma \omega(r, t) \cdot rB_{R/2}(p_i r)\, dr \tag{15}$$

where

$$B_{R/2}(p_i r) = J_{R/2}(p_i r)\, Y_{R/2}(p_i) - Y_{R/2}(p_i r)\, J_{R/2}(p_i) \tag{16}$$

$J_{R/2}(p_i r)$, $Y_{R/2}(p_i r)$ are the Bessel functions of first and second kind respectively of order $R/2$ and p_i is a positive root of the equation.

$$J_{R/2}(p_i \sigma)\, Y_{R/2}(p_i) - Y_{R/2}(p_i \sigma)\, J_{R/2}(p_i) = 0 \tag{17}$$

Applying the finite Hankel transform define by Eqn. (15) and using the boundary conditions (14), the Eqn. (12) transforms to

$$\frac{d\bar{\omega}}{dt} + (p_i^2 + m^2)\,\bar{\omega} = \frac{2}{\pi}\left[\frac{J_{R/2}(p_i)}{J_{R/2}(p_i\sigma)}\,\sigma^{-R/2}\,\phi_2(t) - \phi_1(t)\right] \tag{18}$$

and the initial condition (13) transforms to

$$\bar{\omega} = 0 \text{ at } t = 0 \tag{19}$$

Solution of Eqn. (18) under Eqn. (19) is

$$\bar{\omega} = \frac{2}{\pi} \int_0^t \left[\frac{J_{R/2}(p_i)}{J_{R/2}(p_i\sigma)} \sigma^{-R/2} \phi_2(\tau) - \phi_1(\tau) \right] \exp\left(-\left(p_i^2 + m^2\right)(t-\tau)\right) d\tau \tag{20}$$

Using the inversion formula for the Hankel transform, Tranter[5] and the Eqn. (11), we get

$$u(r, t) = \pi \sum_{i=1}^{\infty} \frac{r^{R/2} p_i^2 J^2_{R/2}(p_i\sigma) B_{R/2}(p_i r)}{J^2_{R/2}(p_i) - J^2_{R/2}(p_i\sigma)} \times \int_0^t \left[\frac{J_{R/2}(p_i)}{J_{R/2}(p_i\sigma)} \sigma^{-R/2} \phi_2(\tau) - \phi_1(\tau) \right] \exp\left(-\left(p_i^2 + m^2\right)(t-\tau)\right) d\tau \tag{21}$$

the summation being over the positive roots of Eqn. (17), Eqn. (21) represents the most general solution of the problem.

Special Case

Cylinders moving with exponentially time dependent velocities. In this case let

$$\phi_1(t) = u_1 e^{-\lambda_1 t} \text{ and } \phi_2(t) = u_2 e^{-\lambda_2 t}, \tag{22}$$

where u_1, u_2, λ_1 and λ_2 are constants.
Substituting Eqn. (22) in Eqn. (21) and on simplifying, we get

$$u(r, t) = \pi \sum_{i=1}^{\infty} \frac{r^{R/2} p_i^2 J^2_{R/2}(p_i\sigma) B_{R/2}(p_i r)}{J^2_{R/2}(p_i) - J^2_{R/2}(p_i\sigma)} \times \left[\left\{ \frac{J_{R/2}(p_i)}{J_{R/2}(p_i\sigma)} \times \frac{u_2 \sigma^{-R/2} e^{-\lambda_2 t}}{(p_i^2 + m^2 - \lambda_2)} - \frac{u_1 e^{-\lambda_1 t}}{(p_i^2 + m^2 - \lambda_1)} \right\} - \left\{ \frac{J_{R/2}(p_i)}{J_{R/2}(p_i\sigma)} \times \frac{u_2 \sigma^{-R/2}}{(p_i^2 + m^2 - \lambda_2)} - \frac{u_1}{(p_i^2 + m^2 - \lambda_1)} \right\} \times \exp\left(-\left(p_i^2 + m^2\right)\right) t \right] \tag{23}$$

To express Eqn. (23) in further simplified form, we note that the Fourier-Bessel expansion for a function $f(r)$ over the range 1 to σ is given by, Tranter[6]

$$f(r) = \sum_{i=1}^{\infty} A_i B_{R/2}(p_i r) \tag{24}$$

where the summation is over the positive roots of Eqn. (17) and the coefficients A_i are given by

$$A_i = \frac{\pi^2 p_i^2 J_{R/2}^2 (p_i \sigma)}{2\{J_{R/2}^2 (p_i) - J_{R/2}^2 (p_i \sigma)\}} \left\{ \int_1^\sigma f(r) \cdot r\, B_{R/2} (p_i r)\, dr \right\} \tag{25}$$

Now expressing each term on R.H.S. of the following expressions in terms of Fourier-Bessel expansion (24) and making use of Eqn. (17), it can be shown that

$$\pi \sum_{i=1}^{\infty} \frac{p_i^2 J_{R/2}(p_i) J_{R/2}(p_i\sigma) B_{R/2}(p_i r)}{\{ J_{R/2}^2 (p_i) - J_{R/2}^2 (p_i\sigma)\} \{p_i^2 + m^2 - \lambda_2\}}$$

$$= \frac{K_{R/2}\{(m^2 - \lambda_2)^{1/2} r\} I_{R/2}\{m^2 - \lambda_2)^{1/2}\} - K_{R/2}\{(m^2 - \lambda_2)^{1/2}\} \cdot I_{R/2}\{(m^2 - \lambda_2)^{1/2} r)\}}{I_{R/2}\{(m^2 - \lambda_2)^{1/2}\} K_{R/2}\{(m^2 - \lambda_2)^{1/2} \sigma\} - I_{R/2}\{(m^2 - \lambda_2)^{1/2} \sigma\} \cdot K_{R/2}\{(m^2 - \lambda_2)^{1/2}\}} \tag{26}$$

and

$$\pi \sum_{i=1}^{\infty} \frac{p_i^2 J_{R/2}^2 (p_i\sigma) B_{R/2}(p_i r)}{\{J_{R/2}^2 (p_i) - J_{R/2}^2 (p_i\sigma)\} \{\lambda_1 - m^2 - p_i^2 \}}$$

$$= \frac{K_{R/2}\{(m^2 - \lambda_1)^{1/2}\} I_{R/2}\{(m^2 - \lambda_1)^{1/2} r\} - I_{R/2}\{(m^2 - \lambda_1)^{1/2} \sigma\} \cdot K_{R/2}\{(m^2 - \lambda_1)^{1/2} r\}}{I_{R/2}\{(m^2 - \lambda_1)^{1/2}\} K_{R/2}\{(m^2 - \lambda_1)^{1/2} \sigma\} - I_{R/2}\{(m^2 - \lambda_1)^{1/2} \sigma\} \cdot K_{R/2}\{(m^2 - \lambda_1)^{1/2}\}} \tag{27}$$

In view of Eqns. (26) and (27), Eqn. (23) can be written as

$$u(r, t) = u_2 \exp\left(-\lambda_2 t \left(\frac{r}{\sigma}\right)^{R/2}\right)$$

$$\times \left[\frac{K_{R/2}\{(m^2 - \lambda_2)^{1/2} r\} I_{R/2}\{(m^2 - \lambda_2)^{1/2}\} - K_{R/2}\{(m^2 - \lambda_2)^{1/2}\} \cdot I_{R/2}\{(m^2 - \lambda_2)^{1/2} r\}}{I_{R/2}\{(m^2 - \lambda_2)^{1/2}\} K_{R/2}\{(m^2 - \lambda_2)^{1/2} \sigma\} - I_{R/2}\{(m^2 - \lambda_2)^{1/2} \sigma\} \cdot K_{R/2}\{(m^2 - \lambda_2)^{1/2}\}} \right]$$

$$- u_1 r^{R/2} \exp(-\lambda_1 t)$$

$$\times \left[\frac{K_{R/2}\{(m^2-\lambda_1)^{1/2}\sigma\} I_{R/2}\{(m^2-\lambda_1)^{1/2} r\} - I_{R/2}\{(m^2-\lambda_1)^{1/2}\sigma\} \cdot K_{R/2}\{(m^2-\lambda_1)^{1/2} r\}}{I_{R/2}\{(m^2-\lambda_1)^{1/2}\} K_{R/2}\{(m^2-\lambda_1)^{1/2}\sigma\} - I_{R/2}\{(m^2-\lambda_1)^{1/2}\sigma\} \cdot K_{R/2}\{(m^2-\lambda_1)^{1/2}\}} \right]$$

$$- \pi \sum_{i=1}^{\infty} \frac{r^{R/2} p_i^2 J_{R/2}^2 (p_i\sigma) B_{R/2}(p_i r)}{J_{R/2}^2(p_i) - J_{R/2}^2(p_i\sigma)}$$

$$\times \left[\left\{ \frac{J_{R/2}(p_i)}{J_{R/2}(p_i\sigma)} \cdot \frac{u_2\, \sigma^{-R/2}}{p_i^2 + m^2 - \lambda_2} - \frac{u_1}{p_i^2 + m^2 - \lambda_1} \right\} \right.$$

$$\left. \times \exp\left(-\left(p_i^2 + m^2\right) t\right) \right] \tag{28}$$

The flow, when inner cylinder is at rest and outer is moving with constant velocity, can readily be obtained by letting u_1 and λ_2 to be zero in the above equation. Thus substituting $u_1 = 0 = \lambda_2$ in Eqn. (28), we get

$$u(r, t) = u_2 \left(\frac{r}{\sigma}\right)^{R/2} \left[\frac{K_{R/2}(mr)\, I_{R/2}(m) - K_{R/2}(m)\, I_{R/2}(mr)}{I_{R/2}(m)\, K_{R/2}(m\sigma) - I_{R/2}(m\sigma)\, K_{R/2}(m)} \right]$$

$$- \pi u_2 \sum_{i=1}^{\infty} \frac{r^{R/2}\, p_i^2\, J_{R/2}(p_i)\, J_{R/2}(p_i\sigma)\, B_{R/2}(p_i r) \exp\left(-\left(p_i^2 + m^2\right) t\right)}{\sigma^{R/2}\left(p_i^2 + m^2\right)\left\{J_{R/2}^2(p_i) - J_{R/2}^2(p_i\sigma)\right\}} \tag{29}$$

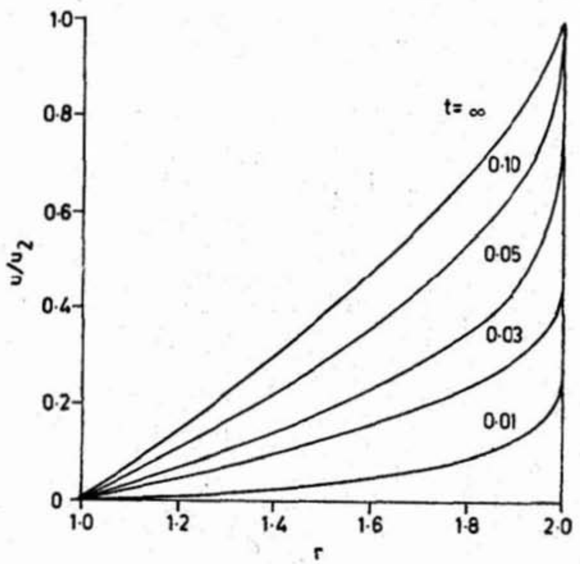

Figure 1. Velocity profiles when outer cylinder is moving for $m = 4$ and $R = 0$.

Equation (29) determines the velocity of the fluid when outer cylinder is moving with constant velocity u_2 and inner cylinder is at rest.

Discussion

Equation (21) gives the general value of the velocity of the fluid for all values of cross-flow Reynolds number and Hartmann number. From this various particular cases

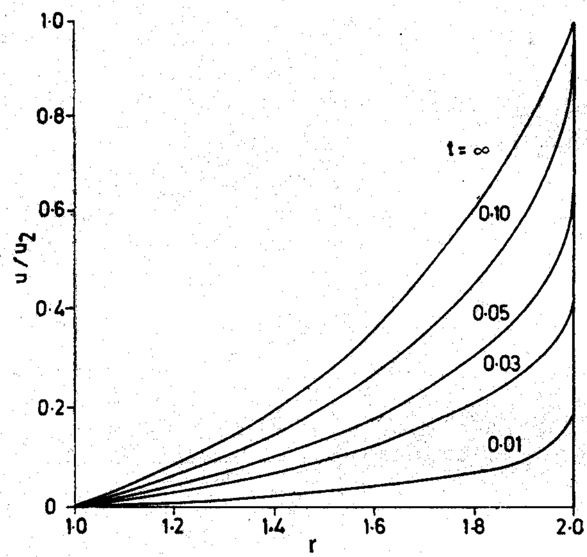

Figure 2. Velocity profiles when outer cylinder is moving for $m = 4$ and $R = 2$.

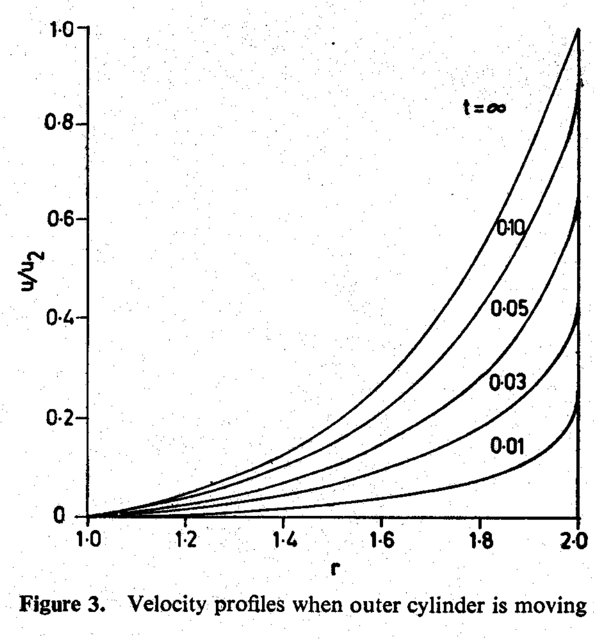

Figure 3. Velocity profiles when outer cylinder is moving for $m = 4$ and $R = 4$.

can easily be obtained for several values of $\phi_1(t)$ and $\phi_2(t)$ i.e. when either inner or outer or both the cylinders are given particular type of velocities. It should be noted that positive R means that there is injection of the fluid at the inner cylinder and suction at the outer cylinder and vice versa for negative values of R.

Equation (28) gives the general value of the velocity of the fluid when cylinders are moving with exponentially time dependent velocities. From this, it is obvious that the velocity decreases according to increase in time and becomes zero when t tends to infinity.

As a special case of Eqn. (28), when the inner cylinder is at rest and the outer is moving with constant velocity, the value of the velocity profiles is calculated which is given in Eqn. (29) and these values of velocity profiles are plotted against various values of R in Figs. 1 to 4. From these figures it is observed that with increase in cross-flow Reynold's number R the velocity decreases. Also it is evident from these figures that the velocity increases with time and ultimately approaches to the steady state.

Further, from Eqns. (28) and (29), it can readily be verified that with increase in Hartmann number m, the velocity decreases.

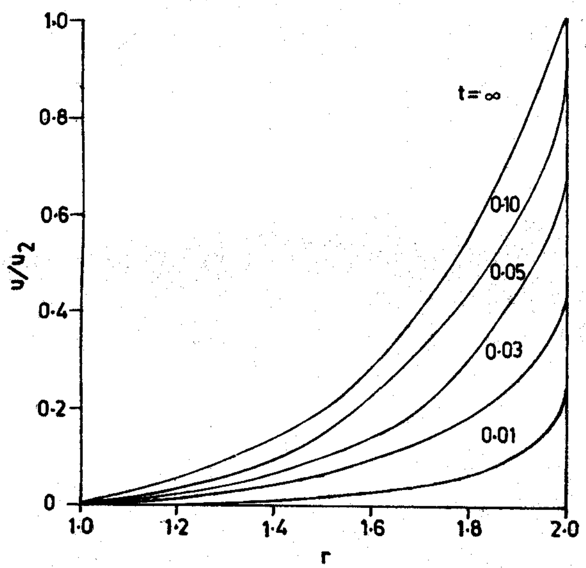

Figure 4. Velocity profiles when outer cylinder is moving for $m = 4$ and $R = 6$.

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