Def Sci J, Vol 31, No 1, January 1981, pp 39-44

On Unsteady Flow of a Dissociating Gas

ASHOK KUMAR

Department of Mathematics, University of Gorakhpur, Gorakhpur-273001

Received 19 October 1979; revised 28 January 1980

Abstract. Lighthill's model of an ideal dissociating gas has been exploited to study three dimensional unsteady flows in the absence of dissipative mechanisms. Expressions for the tangent, principal normal and binormal vectors, curvature and torsion of streamlines, variations of pressure along streamlines and their principal normal and binormal have been calculated and dissociation effects have been studied.

Introduction

It is well known that in the temperature range 1000 to 7000°K dissociation is important while electronic excitation energy and ionization can be neglected. It is not possible to study the phenomena of dissociation in all its generality because of its complicated behaviour. Lighthill[1], however, has introduced the dynamics of a dissociating gas through an idealized model (known since then as Lighthill's model) which has proved to be very useful in various estimates of shock wave phenomena. The different aspects of shock waves in the case of inert gases have been thoroughly discussed by Thomas and others[2-7] who have studied a variety of problems regarding shock waves. In the present paper, we propose to exploit Lighthill's model to study three dimensional unsteady dissociating gas flows in the absence of dissipative mechanisms. The expressions for the tangent, principal normal and binormal vectors and the curvature and torsion of the streamline have been obtained in terms of the velocity components, pressure, density and the mass fraction variable. Variations of pressure along the streamlines and their principal normal and binormal are also obtained. It is shown that the dissociating character of a gas in the case of an unsteady flow is to decrease the pressure gradient considerably along the streamline. Also, from the relation connecting the vorticity components with gradients of dissociation variable and entropy of the system, it is shown that these components are slightly increased in unsteady flow as compared with the case of steady gas flows.

Equations of Motion

The fundamental equations of motion for an unsteady, three dimensional ideal dissociating gas flow referred to a system of rectangular coordinates x_i are[1,6,9], *continuity*

$$\frac{\partial \rho}{\partial t} + \rho_{,i} u_i + \rho u_{j,j} = 0 \tag{1}$$

momentum

$$\rho \frac{\partial u_i}{\partial t} + \rho u_{i,j} u_j + p_{,i} = 0 \tag{2}$$

energy

$$\rho \frac{\partial h}{\partial t} + \rho u_i h_{,i} - \frac{\partial p}{\partial t} - u_i p_{,i} = 0 \tag{3}$$

state

$$p = (1 + q)\, \rho\, RT. \tag{4}$$

$$h = (4 + q)\, RT + Dq \tag{5}$$

reaction rate

$$\frac{\partial q}{\partial t} + q_{,i} u_i = W \tag{6}$$

where ρ, u_i, p, h, q, R, T and D are density, components of the velocity, pressure, enthalpy, mass fraction, gas constant, temperature and dissociation energy per unit mass respectively. Also, the quantity W is given by

$$W = C\rho T^{-n} \left[(1 - q)\, e^{-D/RT} - \frac{\rho}{\rho_D} q^2 \right]$$

where C is a constant, ρ_D is characteristic density and a comma (,) in the above equations as well as in the following indicates partial differentiation with respect to x_i. The internal energy e for the ideal dissociating gas is expressed[1] as,

$$e = 3RT + qD \tag{7}$$

where qD is the energy absorbed by dissociation.

Let

$$h_0 = \tfrac{1}{2} v^2 + h \tag{8}$$

where $v^2 = u_i u_i$ and h_0 is the stagnation enthalpy. Obviously, in the case of unsteady flow the total enthalpy is not constant along the streamline. As a consequence of (4) and (5) we have

$$h = \frac{4 + q}{1 + q} \frac{p}{\rho} + Dq \tag{9a}$$

so that, in general, we can write

$$h = h(p, \rho, q). \tag{9b}$$

Using (9) with (1) to (6), we get

$$u_{i,j}u_iu_j - a_f^2\, u_{j,j} + u_i \frac{\partial u_i}{\partial t} = \frac{1+q}{3}\left(\frac{\partial h}{\partial t} - \frac{1}{\rho}\frac{\partial \rho}{\partial t}\right) + \frac{a_f^2}{\rho}\frac{\partial p}{\partial t}$$

$$+ \left[\frac{D(1+q)}{3} - \frac{3a_f^2}{(4+q)(1+q)}\right] u_i q_{,i} \tag{10}$$

where a_f is the 'frozen' speed of sound given by

$$a_f^2 = \frac{\gamma p}{\rho} = \frac{4+q}{3}\frac{p}{\rho}$$

γ being a function of q.

Variation of Pressure and its Gradients Along and Perpendicular to the Stream Lines

Let λ_i, μ_i and ν_i be respectively the unit tangent, principal normal and binormal vectors at a point on the streamline where its curvature is K. Then, with the help of the well-known Frenet formulae and the Eqns. (2) and (10), we have

$$\lambda_i = \frac{u_i}{v} \tag{11}$$

$$\mu_i = -\frac{1}{K}\left[\frac{\rho\frac{\partial u_i}{\partial t} + p_{,i}}{\rho v^2} + \frac{a_f^2}{v^3}\left\{u_{j,j} - \frac{1}{a_f^2}\, u_r \frac{\partial u_r}{\partial t}\right.\right.$$

$$+ \frac{1+q}{3a_f^2}\left(\frac{\partial h}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t}\right) + \frac{1}{\rho}\frac{\partial \rho}{\partial t}$$

$$\left.\left. + \left(\frac{D\overline{1+q}}{3a_f^2} - \frac{3}{\overline{4+q}\;\overline{1+q}}\right) q_{,r}u_r\right\}\lambda_i\right] \tag{12}$$

$$\nu_i = -\frac{1}{K\rho v^3}\, e_{ijk}u_j\left(\rho\frac{\partial u_k}{\partial t} + p_{,k}\right), \tag{13}$$

where e_{ijk} is the usual permutation symbol. Further, using (6) and (10) in the equations resulting through the multiplication of (12) by λ_i, μ_i and ν_i, we obtain the following

$$\frac{1}{\rho v}\, p_{,i}\lambda_i = -\frac{1}{v}\frac{\partial u_i}{\partial t}\lambda_i - \frac{1}{M_f^2}\left[u_{j,j} - \frac{1}{a_f^2}\, u_r \frac{\partial u_r}{\partial t}\right.$$

$$+ \frac{(1+q)}{3a_f^2}\left(\frac{\partial h}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t}\right) + \frac{1}{\rho}\frac{\partial \rho}{\partial t} +$$

$$\left\{\frac{D(1+q)}{3a_f^2} - \frac{3}{(4+q)(1+q)}\right\}$$

$$\times \left\{ -\frac{\partial q}{\partial t} + C\rho T^{-n}\left(e^{-D/RT}\,\overline{1-q} - \frac{\rho}{\rho_D}\, q^2 \right)\right\}\Bigg]. \quad (14)$$

$$p_{,i}\mu_i = -\rho \frac{\partial u_i}{\partial t} u_i - K\rho v^2 \quad (15)$$

$$p_{,i}\nu_i = -\rho \frac{\partial u_i}{\partial t} \quad (16)$$

where M_f is the 'frozen' Mach number given by $M_f = \frac{v}{a_f}$. The equation (12) gives the curvature K of the streamline at any point. If τ be the torsion of the streamline, then using Frenet formula

$$\frac{d\nu_i}{ds} = -\tau u_i,$$

in (1), (2), (6) and (10), we get

$$\tau\mu_i = \frac{e_{ijk}}{K^2\rho v^6}\left[\frac{\partial u_k}{\partial t}\left\{K\rho v^2 u_r u_{j,r} - u_j u_r K_{,r}\,\rho v^2\right\} + K v^2 u_j p_{,k}\frac{\partial}{\partial t}(\log\rho) + K\rho v^2 u_r u_j \frac{\partial u_{k,r}}{\partial t}\right]$$

$$+ \frac{e_{ijk}}{K^2\rho v^6}\Bigg[K v^2\left\{u_r u_{j,r} p_{,k} + u_j u_r p_{,kr}\right\} - u_j p_{,k} K_{,r} u_r v^2$$

$$- K u_j\left(-p_{,k}v^2 + 3\left(p_{,k} + \rho\frac{\partial u_k}{\partial t}\right) a_f^2\right) u_{r,r}$$

$$- 3K u_j\left(p_{,k} + \rho\frac{\partial u_k}{\partial t}\right) a_f^2\left\{ -\frac{1}{a_f^2} u_k \frac{\partial u_k}{\partial t} + \frac{1+q}{3a_f^2}\right.$$

$$\times\left(\frac{\partial h}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t}\right) + \frac{\partial}{\partial t}(\log\rho)$$

$$\left. + \left(\frac{\overline{1+q}D}{3a_f^2} - \frac{3}{\overline{4+q}\;\overline{1+q}}\right) q_{,k}u_k\right\}\Bigg] \quad (17)$$

which gives the relationship between the torsion of the streamlines and various parameters of the dissociating gas. Avoiding repetition of the well-known differential geometric results and proceeding exactly on the same lines as in[8], we obtain the pressure gradient $\frac{dp}{ds}$ as,

$$\frac{dp}{ds} = - \rho \frac{\partial u_i}{\partial t} \lambda_i + \frac{\rho v^2(K' + K'')}{M_f^2 - 1} - \frac{\rho v}{(M_f^2 - 1)}$$

$$\left\{ - \frac{1}{a_f^2} u_r \frac{\partial u_r}{\partial t} + \frac{1+q}{3a_f^2} \left(\frac{\partial h}{\partial t} - \frac{1}{\rho} \frac{\partial p}{\partial t} \right) + \frac{1}{\rho} \frac{\partial \rho}{\partial t} \right.$$

$$\left. + \left(\frac{D\overline{1+q}}{3a_f^2} - \frac{3}{\overline{4+q}\,\overline{1+q}} \right) \frac{dq}{ds} v \right\} \tag{18}$$

where K' and K'' are respectively the curvatures of the curves in the congruences determined by the principal normals and binormals to the streamlines. It can be easily seen that the expression for $\frac{dp}{ds}$ given by (18) reduces to that of[8] in the case of a non-dissociating steady gas flow. The expression for pressure gradient shows that the effects of dissociation and unsteadiness in the flow is to decrease the pressure gradient considerably.

Vorticity Components

The vorticity components $\hat{W}_k$ are given by

$$\hat{W}_k = e_{krm} u_{m,r}$$

Then from (4), (5) and (8), we obtain

$$p = \frac{1+q}{2(4+q)} \rho \{(2h_0 - v^2) - 2qD\}$$

which, after differentiation w.r.t. x_i yields,

$$p_{,i} = \frac{p}{\rho} \rho_{,i} + \frac{1}{3\gamma\,\overline{\gamma - 1}} pq_{,i} - \rho \frac{\gamma - 1}{\gamma} q_{,i}$$

$$- \rho \frac{\gamma - 1}{\gamma} (u_{j,i} u_j - h_{0,i}) \tag{19}$$

Further, substituting for $p_{,i}$ from the relation,

$$p = p(\rho, S, q)$$

where S is the entropy of the system and using (2) and (19) we obtain

$$e^{ijk} u_j \hat{W}_k = - \left[\frac{1}{\rho} \frac{\partial p / \partial S}{\overline{\gamma - 1}} S_{,i} + Dq_{,i} \right] + \frac{\partial u_i}{\partial t} + h_{0,i} \tag{20}$$

The equation (20) shows that whereas the effect of dissociation is to decrease the vorticity components as compared to the case of ordinary gas flow, the unsteady flow increases these components slightly.

Acknowledgement

The author is thankful to Prof. B. G. Verma for suggesting the problem and helping in the preparation of this paper.

References

1. Lighthill, M. J., *J. Fluid Mech*, **2** (1957), 1.
2. Thomas, T. Y., *Comm. Pure Appl. Math.*, **3** (1950), 103.
3. Taub, A. H., *Rev. Mod. Phys.*, **21** (1949), 584.
4. Kanwal, R. P., *Quart. Appl. Math.*, **16** (1958), 361.
5. Kanwal, R. P., *Proc. Roy. Soc.*, **257** (1960), 263.
6. Kanwal, R. P., *Arch. Rat. Mech. Anal.*, **1** (1958), 225.
7. Verma, B. G., *Tensor N. S.*, **17** (1966), 74.
8. Kanwal, R. P., *J. Math. Mech.*, **6** (1957), 621.
9. Bhutani, O. P. & Rama Shankar, *Ind. Univ. Math. J.*, **20** (1970), 239.