Def Sci J, Vol 31, No 1, January 1981, pp 31–38

Steady Motion of a Circular Cylinder in a Dusty Gas

E. RUKMANGADACHARI

Department of Mathematics, P. R. Govt. College, Kakinada-533001

Received 24 November 1979

Abstract. Effect of uniform motion of a circular cylinder in the dusty gas otherwise at rest at infinity is discussed. The dust particles are assumed to have small relaxation time. An equation for the mass concentration of dust near the cylinder is derived and solved analytically using a potential solution of gas flow at large Reynolds number. Particles do not collide with the cylinder until the stokes number σ is greater than 1/8, if we assume that the gas flow is unchanged by the presence of dust particles.

Introduction

The study of two-phase motion has wide applications. Michael[1] treating the problem of steady motion of a sphere in a dusty gas based on Saffman's model[2] has given several examples of situations where multiphase motions are involved. A perturbation solution for the problem has been obtained, when the relaxation time τ is small. Michael & Norey[3] have considered slow motion of a sphere in a two-phase medium. Sambasiva Rao[4] has studied the uniform translation of a sphere in a dusty gas on the lines of Michael[1].

In this paper the steady motion in a dusty gas of a circular cylinder is considered. We have obtained closed form solution for mass concentration of dust and derived the critical value of σ, the stokes number, equal to $\tau U/a$, where U is the velocity of the cylinder of radius a and τ is the relaxation time for the dust. Some interesting points regarding the motion of particles on the upstream axis near the cylinder have been discussed.

Formulation

The equations governing the motion of an incompressible dusty gas, as given by Saffman[2] are

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\, \bar{u} = - \frac{1}{\rho} \text{ grad } p + \nu \nabla^2 \bar{u} + \frac{KN}{\rho} (\bar{v} - \bar{u}) \tag{1}$$

$$\text{div } \bar{u} = 0 \tag{2}$$

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\, \bar{v} = \frac{K}{m}\, (\bar{u} - \bar{v}) \tag{3}$$

$$\frac{\partial N}{\partial t} + \operatorname{div} N\bar{v} = 0 \tag{4}$$

Here $\bar{u}$ and $\bar{v}$ are the gas and dust velocity vectors respectively, N is the number density of dust particles, each of mass m, and K is the stokes coefficient of resistance; and p, ρ, μ represent respectively the pressure, density and viscosity of the gas.

The two parameters which influence the motion of dust gas are the time relaxation parameter $\tau = m/K$ and the mass concentration parameter $f = mN/\rho$. When τ is small, which means that the dust is fine and quickly adjusts to the changes in gas velocity, it would seem appropriate to consider perturbation solution for $\tau = 0$ for flow past a circular cylinder at large Reynolds numbers.

We consider the potential flow for the translation of a circular cylinder of radius a with velocity U parallel to x-axis as the limiting case when $\tau = 0$, neglecting the viscous boundary layers and separation effects. Let $\bar{u}_0$ represent the unperturbed velocity of gas and dust. Now

$$\bar{u}_0 = \operatorname{grad} \phi = \frac{Ua^2}{r^2} \cos\theta \hat{r} + \frac{Ua^2}{r^2} \sin\theta \hat{\theta} \tag{5}$$

Therefore

$$\phi = \frac{Ua^2}{r} \cos\theta,\ \psi = -\frac{Ua^2}{r} \sin\theta \tag{6}$$

where (r, θ) are the polar coordinates with $\theta = 0$ as the downstream direction and U is the velocity of translation of the cylinder.

Let $\bar{u} = \bar{u}_0 + \bar{u}'$, $\bar{v} = \bar{v}_0 + \bar{v}'$, $N = N_0 + N'$

$f = f_0 + f'$ and $p = p_0 + p'$

where the primed letters represent small perturbation quantities of order τ.

For the steady flow, the transient part vanishes and from Eqns. (1) and (2), we obtain

$$(\bar{u} \cdot \nabla)\, \bar{u} = -\frac{1}{\rho} \operatorname{grad} p + \nu\nabla^2\bar{u} - f(\bar{v} \cdot \nabla)\, \bar{v} \tag{7}$$

Writing the first order terms, we have from Eqns. (1) and (3)

$$(\bar{u}_0 \cdot \nabla)\, \bar{w}' + \bar{w}' \cdot \nabla u_0 + f(u_0 \cdot \nabla)\, \bar{u}_0 = -\frac{1}{\rho} \operatorname{grad} p' \tag{8}$$

$$\tau\, (\bar{u}_0 \cdot \nabla)\, \bar{u}_0 = \bar{u}' - \bar{v}' \tag{9}$$

where $\bar{w}' = \bar{u}' + f_0\bar{v}'$ and we have neglected the viscous terms. Equation (4) gives

$$f_0 \operatorname{div} \bar{v}' + (\bar{u}_0 \cdot \nabla) f' = 0 \tag{10}$$

using Eqn. (2). Also we obtain from Eqn. (2)

$$(\bar{u}_0 \cdot \nabla) f' = f_0 \tau \nabla^2 \tfrac{1}{2} \bar{u}_0^2 \tag{11}$$

where we have used $(\bar{u}_0 \cdot \nabla) \bar{u}_0 = \text{grad} \tfrac{1}{2} \bar{u}_0^2$.

Having found f' from Eqn. (11), the velocity vectors $\bar{u}'$ and $\bar{v}'$ may be obtained thus. Since we have div $\bar{u}' = 0$ we can write $\bar{u}' = \text{curl}\, \bar{A}$, where $\bar{A}$ has one component $A(r, \theta)$ perpendicular to (r, θ) plane. Then from Eqn. (9)

$$\bar{v}' = \text{curl}\, \bar{A} - \tau (\bar{u}_0 \cdot \nabla) \bar{u}_0 \tag{12}$$

and

$$\bar{w}' = (1 + f_0) \text{curl}\, \bar{A} - f_0 \tau \, \text{grad} \tfrac{1}{2} \bar{u}_0^2.$$

Equation (8) now becomes

$$\begin{aligned}(\bar{u}_0 \cdot \nabla) \{(1 + f_0) \text{curl}\, \bar{A} - f_0 \tau \, \text{grad} \tfrac{1}{2} \bar{u}_0^2\} \\ + \{(1 + f_0) \text{curl}\, \bar{A} - f_0 \tau \, \text{grad} \tfrac{1}{2} \bar{u}_0^2\} \cdot \nabla \bar{u}_0 \\ + f' \, \text{grad} \tfrac{1}{2} \bar{u}_0^2 = - \frac{1}{\rho} \text{grad}\, p'\end{aligned} \tag{13}$$

This equation gives two component equations, which when solved give $A(r, \theta)$ and p' when f' is known. Equation (13) can be put in the alternate form

$$(1 + f_0) \{(\bar{u}_0 \cdot \nabla) \text{curl}\, \bar{A} + (\text{curl}\, \bar{A} \cdot \nabla) \bar{u}_0\} + f' \, \text{grad} \tfrac{1}{2} \bar{u}_0^2 = - \frac{1}{\rho} \text{grad}\, p'' \tag{14}$$

where $\quad p'' = p' - f_0 \tau (\bar{u}_0 . \, \text{grad} \tfrac{1}{2} \bar{u}_0^2)$.

Now we take up the solution of Eqn. (11). We note that

$$\nabla^2 \tfrac{1}{2} \bar{u}_0^2 = \frac{8 U^2 a^4}{r^6}$$

which is always positive and is independent of θ. Thus f' increases monotonically along a stream line; and the rate of increase is same for all θ.

Written in terms of (r, θ), the Eqn. (11) becomes

$$\cos \theta \frac{\partial f'}{\partial r} + \frac{1}{r} \sin \theta \frac{\partial f'}{\partial \theta} = 8 f_0 \tau U \cdot \frac{a^2}{r^4} \tag{15}$$

This can be put in non-dimensional form

$$\cos \theta \frac{\partial \bar{f}}{\partial r'} + \frac{1}{r'} \sin \theta \frac{\partial \bar{f}}{\partial \theta} = \frac{1}{(r')^4} \tag{16}$$

where $\quad \bar{f} = \left(\frac{a}{8 f_0 \tau U}\right) f'$ and $r' = r/a$

Integrating, we obtain

$$f' = F(r'/\sin \theta) + \alpha \bar{c}^3 (- \cot \theta - \tfrac{1}{3} \cot^3 \theta) \; (\alpha = 8 f_0 \tau U/a)$$

where $F(r'/\sin\theta)$ is an arbitrary function. Using the boundary condition that at $\theta = \pi/2, f' = 0$, we note that $F(r')$ should vanish for all r', so that $F = 0$. Thus we have

$$f' = -\frac{\alpha}{(r')^3}(\cos\theta\sin^2\theta + \tfrac{1}{3}\cos^3\theta) \tag{17}$$

For $\theta = 0$, $\theta = \pi$, we have the stagnation points. At these points, we have $|\bar{f}| = 1/3$

i.e. $$f' = \frac{8 f_0 \tau U}{3a} \tag{18a}$$

and the corresponding result for the sphere[4] is

$$f' = \frac{27 f_0 \tau U}{8a} \tag{18b}$$

Thus, we see that the additional mass concentration of dust up to $0(\tau)$ in the case of a sphere is $(9/8)^2$ times greater than that in the case of a circular cylinder.

In nondimensional form the unperturbed streamlines are given by

$$r' = c\sin\theta \tag{19}$$

We can write equation (11) as

$$\bar{u}_0\frac{\partial f'}{\partial s} = f_0\tau\nabla^2\tfrac{1}{2}\bar{u}_0^2 \tag{20}$$

where $\partial f'/\partial s$ represents the rate of change of f' with length along a streamline. Using Eqn. (19) to eliminate θ, we may obtain the following expressions for $\partial\bar{f}/\partial r$ and $\partial\bar{f}/\partial\theta$ on the c-streamline, where $\bar{f} = af'/8f_0\tau U$

$$\left(\frac{\partial\bar{f}}{\partial\theta}\right)_{\psi=c} = \frac{c}{r^4};\ \left(\frac{\partial\bar{f}}{\partial r}\right)_{\psi=c} = \pm\frac{c}{r^4\sqrt{c^2 - r^2}} \tag{21}$$

In the expression for $(\partial\bar{f}/\partial r)_{\psi=c}$ the negative sign is to be taken from $\theta = \pi$ to $\theta = \pi/2$ and the positive sign from $\theta = \pi/2$ to $\theta = 0$. Values of $\bar{f}$ could be obtained at various points along the c-streamlines for different values of c by numerical integration using, say, Simpson's $\frac{1}{3}$ rule, but the integration procedure is complicated due to the fact that both the above formulae have to be used on each c-streamline; and we have not attempted it here.

Small Values of f

We assume that f_0 is small. Then Eqn. (10) shows that f' is of the second order of smallness and to the first-order we observe from Eqn. (8) that $\bar{u}' = p = 0$. We have from (9)

$$\bar{v}' = -\tau(\bar{u}_0 \cdot \nabla)\,\bar{u}_0 \tag{22}$$

so that

$$\bar{v} = \bar{u}_0 + \bar{v}' = \text{grad}\,[\phi - \tfrac{1}{2}\,\tau(\text{grad}\,\phi)^2] \tag{23}$$

Thus, in this case, $\bar{v}$ remains to be a potential field and the potential is given in dimensionless form by

$$\Phi = Ua\left(\frac{1}{r}\cos\theta - \tfrac{1}{2}\,\sigma\,\frac{1}{r^4}\right) \tag{24}$$

The equation for dust streamlines is

$$\frac{dr}{r d\theta} = \frac{\cos\theta - \sigma\,\dfrac{2}{r^3}}{\sin\theta} = \cot\theta - \frac{2\sigma}{r^3}\,\text{cosec}\,\theta \tag{25}$$

When $\sigma = 0$, this equation integrates to give Eqn. (19); and it is interesting to trace the divergence of the gas particles from the paths given by Eqn. (19). To this end, we write the equation of the streamline in the form

$$\frac{\sin\theta}{r} = k + k'(\theta) \tag{26}$$

where $k'(\theta)$ is a small change in k of order σ, representing the displacement of the particles at an angle θ. Differentiating (26), we obtain

$$\frac{dk'}{d\theta} = \frac{\cos\theta}{r} - \frac{1}{r^2}\sin\theta\,\frac{dr}{d\theta} \tag{27}$$

Elimination of $\dfrac{dr}{d\theta}$ between (25) and (27) yields

$$\frac{dk'}{d\theta} = \frac{2\sigma}{r^4} \tag{28}$$

This shows an interesting feature. The dust path lines coincide with the gas path lines at a distance far away from the cylinder as in the case of the translation of a sphere[4]. Further, in the case of motion of a sphere the dust path lines and the gas path lines coincide along $\theta = 0$. It is of interest to note that in contrast to this, the path lines of dust and gas do not coincide at small distances for any value of θ in the case of circular cylinder since Eqn. (28) is independent of θ.

Critical Value of σ

We will now find the critical value of σ at which particles begin to collide with the cylinder. In order to do this, we assume that the gas velocity is unchanged by the dust, and that head-on collisions with the cylinder by particles on the upstream axis will be the first to occur.

The equation of motion for a particle on this axis is

$$\frac{dv}{dr} = -\frac{\left(v - \frac{1}{r^2}\right)}{\sigma v} \tag{29}$$

where $v = v_r/U$. We have to solve Eqn. (29) with the boundary condition that $v = 0$ at $r = \infty$.

Let us investigate the behaviour of the solution of Eqn. (29) at the front stagnation point $r = 1$. Writing $r = 1 + h$, where h is small, we have

$$\frac{dv}{dh} = -\frac{v - (1 - 2h)}{\sigma v} \tag{30}$$

neglecting powers of h greater than or equal to two. This equation may be written in parametric form with the parameter t proportional to time,

$$\frac{dv}{dt} = 1 - 2h - v \tag{31}$$

$$\frac{dh}{dt} = \sigma v \tag{32}$$

Thus the complementary function in the solution for v and h has the form $e^{\lambda t}$ where λ satisfies the equation

$$\lambda^2 + \lambda + 2\sigma = 0 \tag{33}$$

In addition to the complementary function, a particular integral $h = \frac{1}{2}$ exists for h, while there is no particular integral for v.

We observe that when $\sigma \leqslant \frac{1}{8}$ the roots λ_1 and λ_2 of Eqn. (33) are real and negative. When $\sigma > \frac{1}{8}$, λ_1 and λ_2 are complex conjugates.

An interesting feature noticed in the case of translation of a circular cylinder is that if $\sigma \leqslant \frac{1}{8}$ and when $h < \frac{1}{2}$ initially the dust particles move away from the cylinder and tend to reach the point $h = \frac{1}{2}$ and when $h > \frac{1}{2}$ initially the dust particles move towards the cylinder and tend to reach the same point, $h = \frac{1}{2}$; and the time taken to reach it approaches infinity like log h as $h \to \frac{1}{2}$, in either case. Thus, the dust particles on either side of the point $h = \frac{1}{2}$ continue to remain on the same side and they never reach the body even after a long time unlike in the case of flow past a body. Similar behaviour is noticeable in the case of translation of sphere also, where the corresponding point is $h = \frac{1}{3}$, but Sambasiva Rao[4] has mistakenly concluded that when $\sigma \leqslant \frac{1}{12}$, the time taken for the particles to come to stagnation point approaches infinity like log h as $h \to 0$.

When $\sigma > \frac{1}{8}$, we find that v is non-zero for $h = 0$ and in this case, the dust particles collide with the cylinder in finite time. This is similar to the behaviour of particles in the case of sphere reported by Michael[1].

It may also be noted that the critical value of σ for the cylinder, which is $\frac{1}{8}$, is $\frac{3}{2}$ times greater than that for the sphere viz. $\frac{1}{12}$.

It is interesting to note that there is marked difference in the behaviour of dust particles in the two cases : (i) when a body translates in infinite dusty fluid and (ii) when the dusty fluid is flowing past a stationary body. The differential equation for the mass-concentration f' of dust has been found by Michael[1] to be such that it could not be integrated in a simple form in general, in the case of flow past the sphere. But, considering the problem of steady motion of a sphere in dusty gas, Sambasiva Rao[4] has found the solution in closed form for the corresponding equation. Similar features are noticed by the author for the motion of a circular cylinder.

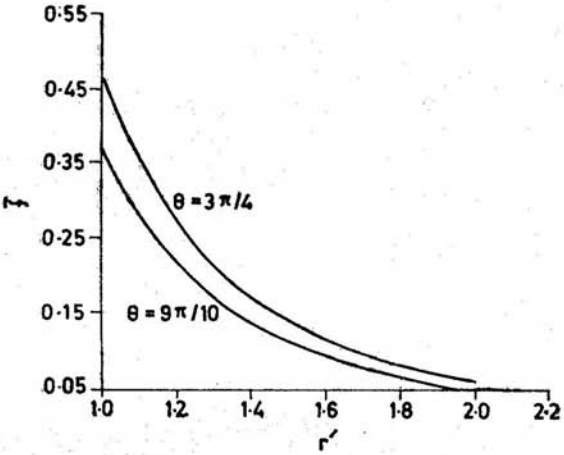

Figure 1. Mass concentration of dust $\bar{f}$ plotted against radial distance r' for $\theta = \frac{3\pi}{4}$ and $\frac{9\pi}{10}$ $(r' = r/a)$.

Table 1. Values of mass-concentration of dust $\bar{f} = \frac{1}{(r')^3}(2/3\cos^2\theta - 1)\cos\theta$ for various values of r' and θ.

r' \ $\theta =$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
1.0	0.0	0.34532164	0.47140452	0.39815920	0.33333333
1.1	0.0	0.25944526	0.35417319	0.29914290	0.25043824
1.2	0.0	0.19983891	0.27280354	0.23041620	0.19290123
1.3	0.0	0.15717872	0.21456737	0.18122858	0.15172204
1.4	0.0	0.12584608	0.17179465	0.14510175	0.12147715
1.5	0.0	0.10231752	0.13967541	0.11797310	0.09876543
1.6	0.0	0.08430704	0.11508899	0.09720683	0.08138021
1.7	0.0	0.07028733	0.09595044	0.08104197	0.06784721
1.8	0.0	0.05921153	0.08083068	0.06827147	0.05715592
1.9	0.0	0.05034577	0.06872788	0.05804916	0.04859795
2.0	0.0	0.04316520	0.05892556	0.04976990	0.04166666

Numerical Work

In Fig. 1, mass concentration of dust $\bar{f}$ is plotted against the radial distance r' for $\theta = 3\pi/4$ and $9\pi/10$. It shows that the curve falls off as we move away from the cylinder. Close to the cylinder i.e. from $r' = 1.0$ to $r' = 1.2$ the fall is rather steep when compared with the situation at a greater distance. Table 1 gives the values of $\bar{f}$ for a range of values of r' and θ. It is observed that $\bar{f}$ has a maximum value for $\theta = 3\pi/4$ or $\pi/4$.

References

1. Michael, D. H., *J. Fluid Mech.*, **31** (1968), 175.
2. Saffman, P. G., *J. Fluid Mech.*, **13** (1962), 120.
3. Michael, D. H. & Norey, P. W., *Can. J. Phy.*, **48** (1970), 1607.
4. Sambasiva Rao, P., *Def. Sci. J.*, **23** (1973), 109.