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Non-Slender Shapes of Minimum Ballistic Factor via Gradient Technique

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Abstract. The problem of finding geometries of minimum re-entry time missile has been reduced to a problem in optimum control and the first order gradient method is applied to obtain Non-Slender shapes of minimum ballistic factor for the cases when (*i*) the diameter and surface area are prescribed, and (*ii*) the length and surface area are prescribed.

Introduction

Earlier there have been attempts to give through variational calculus analystical solutions to the problem of determining slender missile shapes of minimum ballistic factor in hypersonic flow regime under different constraints on the geometrical quantities of the missile, viz., length, diameter and surface area[1–4]. But there are situations where analytical solutions may not be easy to obtain, e.g., when the body is non-slender and in such cases numerical approach to the problem has to be resorted to. One such procedure which has proved to be a powerful numerical computing tool for the optimization of a function of the final values of the dependent variables is the method of gradient. In a recent paper Tawakley[5] demonstrated the utility of this method for finding geometry of a slender axisymmetric missile of minimum ballistic factor. In another study, Singh and Tawakley[6] applied the gradient method and the Raleigh-Ritz method for obtaining non-slender shapes of minimum ballistic factor for the case when length and diameter of the missile are known in advance. In this paper, the gradient method has been further applied to obtain non-slender shapes of minimum ballistic factor for the cases when (*i*) the diameter and surface area are prescribed, and (*ii*) the length and surface area are prescribed.

Formulation of the Problem

It will first be shown how the problem of finding the missile configuration can be reduced to the form where the method of gradient may be easily applied. Considering the Newtonian flow theory and assuming that the body is at zero angle of attack

(Fig. 1), the expressions for drag, volume and surface area of a non-slender body of length l are given by

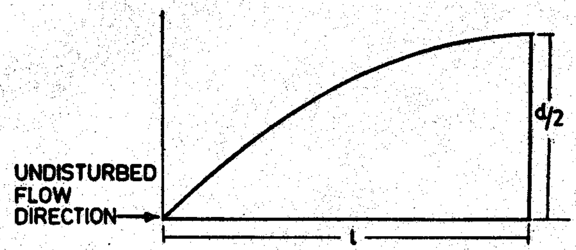

Figure. 1. Coordinate system.

$$\frac{D}{4\pi q} = \int_0^l \frac{yy'^3}{1 + y'^2}\, dx$$

$$\frac{V}{\pi} = \int_0^l y^2\, dx$$

$$\frac{S}{2\pi} = \int_0^l y\sqrt{1 + y'^2}\, dx$$

where $y' = dy/dx$

If we now introduce the dimensionless variables $t = x/l$ and $X_3 = y/l$ and put $y'(x) = u$, the above may be written as

$$\frac{D}{4\pi q l^2} = \int_0^1 \frac{X_3 u^3}{1 + u^2}\, dt$$

$$\frac{V}{\pi l^3} = \int_0^1 X_3^2\, dt$$

$$\frac{S}{2\pi l^2} = \int_0^1 X_3\sqrt{1 + u^2}\, dt$$

Considering t as the independent variable and introducing the following definitions

$$X_1 = \frac{D}{4\pi q l^2},\quad X_2 = \frac{V}{\pi l^3},\quad X_4 = \frac{S}{2\pi l^2}$$

we obtain,

$$\left.\begin{aligned} \dot{X}_1 &= \frac{X_3 u^3}{1+u^2} \\ \dot{X}_2 &= X_3^2 \\ \dot{X}_3 &= u \\ \dot{X}_4 &= x_3\sqrt{1+u^2} \end{aligned}\right\} X_1(0) = X_2(0) = X_3(0) = X_4(0) = 0 \tag{1}$$

Now we are in a position to define our problem in such a way that it may be treated by the gradient method. Taking u as the control variable and X_1, X_2, X_3 and X_4 as the state variables, it is required to find the history of u, i.e., $u = u(t)$, so that the system of equations (1) are satisfied and

Case (*i*) : the quantity $\phi = \dfrac{X_1(1)}{X_2(1)}$ is minimum and the constraint

$$\Psi(1) \equiv X_4(1) - \mu = 0 \tag{2}$$

where

$$\mu = \frac{S}{2\pi l^2}$$

is satisfied.

Case (*ii*) : the quantity $\phi(1) = \dfrac{X_1^2(1)\, X_4(1)}{X_2^2(1)}$ is minimum and the constraint

$$\Psi(1) \equiv \frac{X_4(1)}{X_3^2(1)} - \nu = 0$$

where

$$\nu = \frac{2S}{\pi d^2} \tag{3}$$

is satisfied, d being the diameter of the missile.

Solution by Gradient Method

In [ref. 5] the algorithm of the gradient technique has been described. The same will now be applied to find the minimum ballistic factor shapes for the two cases mentioned above.

Case (*i*) : *Surface area and length are prescribed in advance* — Assuming an initial estimate of $u = u(t)$, the influence function equations corresponding to ϕ and Ψ to be solved together with the state equations (1) are

$$\dot{\lambda}_1^{\phi} = 0$$

$$\dot{\lambda}_2^{\phi} = 0$$

$$\dot{\lambda}_3^{\phi} = -\left[\lambda_1^{\phi} \frac{u^3}{1+u^2} + 2\lambda_2^{\phi} X_3 + \lambda_4^{\phi} \sqrt{1+u^2}\right]$$

$$\dot{\lambda}_4^{\phi} = 0 \tag{4}$$

with the boundary conditions

$$\lambda_1^{\phi}(1) = \frac{1}{X_2(1)}$$

$$\lambda_2^{\phi}(1) = -\frac{X_1(1)}{X_2^2(1)}$$

$$\lambda_3^{\phi}(1) = 0$$

$$\lambda_4^{\phi}(1) = 0$$

and

$$\dot{\lambda}_1^{\psi} = 0$$

$$\dot{\lambda}_2^{\psi} = 0$$

$$\dot{\lambda}_3^{\psi} = -\left[\lambda_1^{\psi} \frac{u^3}{1+u^2} + 2\lambda_2^{\psi} X_3 + \lambda_4^{\psi} \sqrt{1+u^2}\right]$$

$$\dot{\lambda}_4^{\psi} = 0 \tag{5}$$

with the boundary conditions

$$\lambda_1^{\psi}(1) = 0, \quad \lambda_2^{\psi}(1) = 0, \quad \lambda_3^{\psi}(1) = 0, \quad \lambda_4^{\psi}(1) = 1$$

Also the three integrals to be solved together with (4) and (5) are

$$I_{\phi\phi} = \int_0^1 \left\{\lambda_1^{\phi} X_3 \frac{u^2(3+u^2)}{(1+u^2)^2} + \lambda_3^{\phi}\right\}^2 dt$$

$$I_{\psi\psi} = \int_0^1 \left\{\lambda_3^{\psi} + \lambda_4^{\psi} \frac{X_3 u}{\sqrt{1+u^2}}\right\}^2 dt$$

$$I_{\phi\psi} = \int_0^1 \left\{\lambda_1^{\phi} X_3 \frac{u^2(3+u^2)}{(1+u^2)^2} + \lambda_3^{\phi}\right\}\left\{\lambda_3^{\psi} + \lambda_4^{\psi} \frac{X_3 u}{\sqrt{1+u^2}}\right\} dt \tag{6}$$

We now examine the quantity $I_{\phi\phi} I_{\psi\psi} - I^2_{\psi\phi}$. If this quantity approaches zero and the constraint (2) is satisfied then our assumed $u = u(t)$ is the required control variable, otherwise we find a new $u(t)$ given by

$$\bar{u}(t) = u(t) + \delta u(t) \tag{7}$$

where

$$\delta u(t) = a \left\{ \lambda_1^{\phi} \; X_3 \; \frac{u^2(3 + u^2)}{(1 + u^2)^2} + \lambda_3^{\phi} \right\} + b \left\{ \lambda_3^{\psi} + \lambda_4^{\psi} \frac{X_3 u}{\sqrt{1 + u^2}} \right\}$$

the constants a and b are given by

$$\delta_{\phi} = aI_{\phi\phi} + bI_{\phi\psi}$$

$$\delta_{\psi} = aI_{\phi\psi} + bI_{\psi\psi}$$

where $\delta\phi$ and $\delta\psi$ are known changes in ϕ and ψ respectively. With the new $u(t)$ given by (7) we repeat the whole procedure described above till we get $I_{\phi\phi}I_{\psi\psi} - I^2_{\psi\phi} \rightarrow 0$. Following this procedure we obtain the relation between X_3 and t, i.e. the shape of the missile for known values of μ as shown in Fig. 2. Also Fig. 3 illustrates the values of the ballistic factor $X_1(1)/X_2(1)$ ($= Dl/4qV$) for the given values of μ.

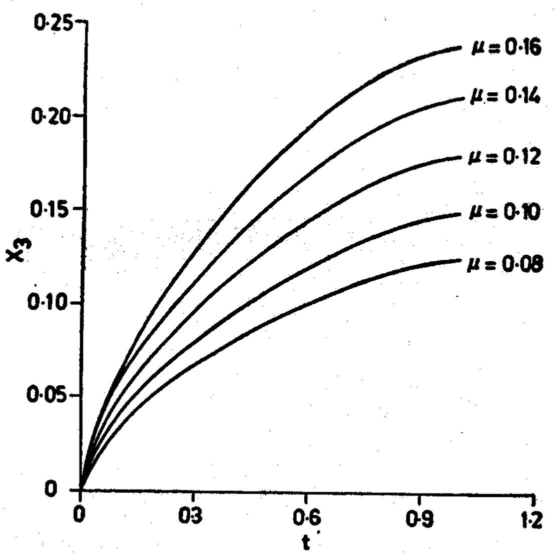

Figure 2. Shapes when S and 1 are prescribed.

Case (ii) : Surface area and diameter are known in advance—With an initial estimate of the control function $u = u(t)$, the influence function equations corresponding to ϕ and ψ to be solved together with the state equations are the same as equations (4) and (5) respectively but with the changed boundary conditions as

$$\lambda_1^{\phi}(1) = \frac{2X_1(1)X_4(1)}{X_2^2(1)}$$

$$\lambda_2^{\phi}(1) = -\frac{2X_1^2(1)X_4(1)}{X_2^3(1)}$$

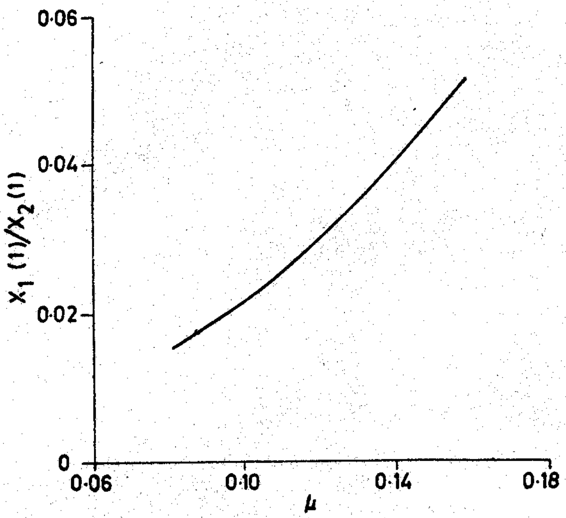

Figure 3. Ballistic factor $X_1(1)/X_2(1)$ vs. $\mu \left(=\frac{S}{2\pi l^2}\right)$.

$$\lambda_3^{\phi}(1) = 0$$

$$\lambda_4^{\phi}(1) = \frac{X_1^2(1)}{X_2^2(1)}$$

and

$$\lambda_1^{\psi}(1) = 0, \quad \lambda_2^{\psi}(1) = 0, \quad \lambda_3^{\psi} = -\frac{2X_4(1)}{X_3^3(1)}, \quad \lambda_4^{\psi} = \frac{1}{X_3^2(1)}$$

Further the integrals corresponding to (6) are

$$I_{\phi\phi} = \int_0^1 \left\{\lambda_1^{\phi} X_3 \frac{u^2(3+u^2)}{(1+u^2)^2} + \lambda_3^{\phi} + \lambda_4^{\phi} X_3 \frac{u}{1+u^2}\right\}^2 dt$$

$$I_{\psi\psi} = \int_0^1 \left\{\lambda_1^{\psi} X_3 \frac{u^2(3+u^2)}{(1+u^2)^2} + \lambda_3^{\psi} + \lambda_4^{\psi} X_3 \frac{u}{1+u^2}\right\}^2 dt$$

$$I_{\phi\psi} = \int_0^1 \left\{\lambda_1^{\phi} X_3 \frac{u^2(3+u^2)}{(1+u^2)^2} + \lambda_3^{\phi} + \lambda_4^{\phi} X_3 \frac{u}{1+u^2}\right\}$$

$$\times \left\{\lambda_1^{\psi} X_3 \frac{u^2(3+u^2)}{(1+u^2)^2} + \lambda_3^{\psi} + \lambda_4^{\psi} X_3 \frac{u}{1+u^2}\right\} dt \qquad (8)$$

As in the earlier case we examine the quantity $I_{\phi\phi} I_{\psi\psi} - I^2_{\psi\phi}$. If it approaches zero and the constraint (3) is satisfied then our assumed $u = u(t)$ is correct, otherwise we improve our solution by taking a new $u(t)$ given by

$$\bar{u}(t) = u(t) + \delta u(t) \tag{9}$$

where

$$\delta u(t) = a\left\{\lambda_1^{\phi} X_3 \frac{u^2(3+u^2)}{(1+u^2)^2} + \lambda_3^{\phi} + \lambda_4^{\phi} X_3 \frac{u}{1+u^2}\right\} + b\left\{\lambda_1^{\psi} X_3 \frac{u^2(3+u^2)}{(1+u^2)^2} + \lambda_3^{\psi} + \lambda_4^{\psi} X_3 \frac{u}{1+u^2}\right\}$$

where a and b are as before obtained from

$$\delta_{\phi} = aI_{\phi\phi} + bI_{\phi\psi}$$

$$\delta_{\psi} = aI_{\phi\psi} + bI_{\psi\psi}$$

δ_{ϕ} and δ_{ψ} being asked for changes in ϕ and ψ respectively. With the new $u(t)$ given by (9) the procedure is repeated till $I_{\phi\phi}I_{\psi\psi} - I^2_{\psi\phi} \longrightarrow 0$. In this way we obtain the

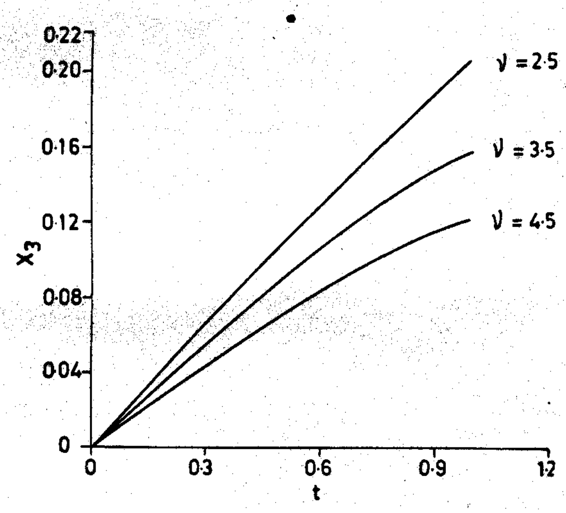

Figure 4. Shapes when S and d are known.

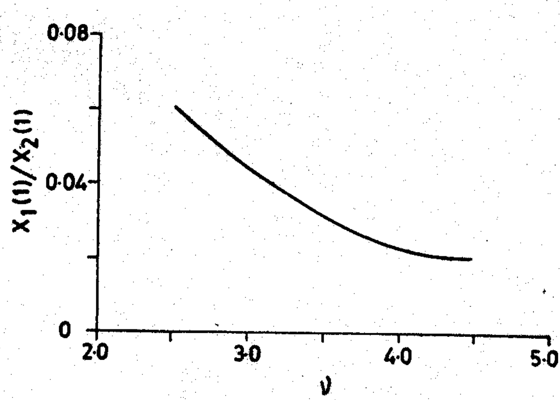

Figure 5. Ballistic factor $X_1(1)/X_2(1)$ vs. $\nu\left(=\frac{2S}{\pi d^2}\right)$.

relation between X_3 and t, i.e., the geometry of the missile for known values of ν as illustrated in Fig. 4. As in case (i), we plot the values of $X_1(1)/X_2(1)$ for given values of ν as given in Fig. 5, and also Fig. 6 represents $X_4(1)$ ($= S/2\pi l^2$) as a function of the quantity ν.

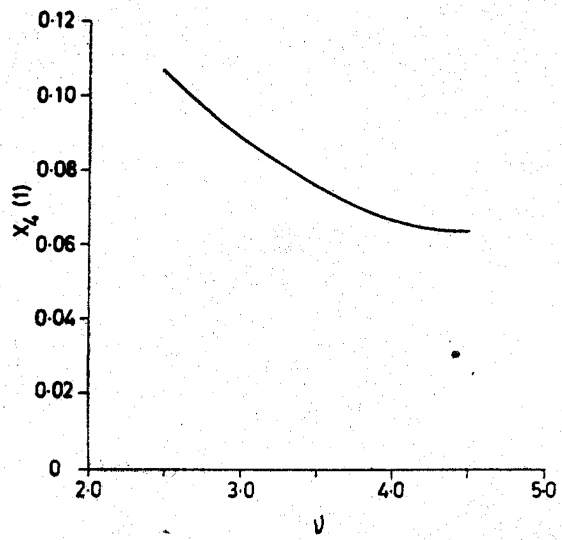

Figure 6. Ballistic Factor $X_4(1)$ vs. ν.

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