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Imploding Detonation Waves

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Abstract. The problem of imploding detonation waves propagating through a gas with varying initial density, is studied. It is shown that the consideration of varying initial density affects the problem considerably incomparison to a uniform gas at rest. An analytical expression for the pressure distribution in the neighbourhood of the centre of symmetry has been found.

Introduction

The problem of imploding cylindrical or spherical shock front propagating into a uniform gas at rest is well known[1,2]. Nigmatulin[3], Welsh[4] and Teipel[5] have replaced the shock front by a contracting detonation front propagating into a uniform combustible gas. If the shock wave is replaced by a detonation wave, similarity solutions cannot be obtained for a general energy release. It can be used only for studying the flow field, if the detonation front is governed by the Chapman-Jouguet conditions. The constant amount of heat is produced during the detonation process and by adding this, the basic flow equations only be corrected.

We have considered the problem of detonation waves into a gas of varying density and only the case at the Chapman-Jouguet point is of interest. The expression for similarity exponent is determined and comparisons have been made with the values for the detonation waves into a uniform gas at rest.

Flow Equations and Conditions at the Detonation Front

We assume that the undisturbed density ρ_1 ahead of the detonation front is given by

$$\rho_1 = {}_0 r^{-k} \tag{1}$$

where ρ_0 and k are positive constants and r is the distance measured from the axis (or centre) of symmetry. The basic equations for the unsteady flow, neglecting viscosity and heat conduction effects, are

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial r} + \sigma \frac{\rho v}{r} = 0 \tag{2}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \tag{3}$$

$$\frac{\partial s}{\partial t} + v \frac{\partial s}{\partial r} = 0 \tag{4}$$

where ρ, v, p, s are density, velocity, pressure, and specific entropy respectively and $\sigma = 0, 1, 2$ are related to the plane, cylindrical and spherical case respectively.

The heat release q per unit mass of gas, for the unsteady flow following Welsh[4] and Teipel[5], is given by

$$\frac{p}{C_p T_1} = \frac{1}{2} \frac{U^2}{C_1^2} \left[1 - 2 \frac{C_1^2}{U^2} + \frac{C_1^4}{U^4} \right] \tag{5}$$

where T, C_1, U, C_p are temperature, sound speed, detonation front velocity and specific heat of constant pressure respectively. Converging cylindrical or spherical detonation waves become very strong near the point of symmetry. Hence the Eqn. (5) can be simplified by retaining only the largest term. From the conditions for the conservations of mass, momentum and energy at the strong detonation wave, we have

$$\frac{p_2}{p_1} = \frac{\gamma}{\gamma + 1} \frac{U^2}{C_1^2} \tag{6a}$$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma} \tag{6b}$$

$$v_2 = \frac{1}{\gamma + 1} U \tag{6c}$$

where γ denotes the ratio of specific heats. The suffixes 2 and 1 refer to conditions just behind and just ahead of the detonation front respectively.

We introduce here the similarity transformations for obtaining the solution so that all the physical quantities depend on

$$\eta = \frac{r}{(-C_1 t)^n} \tag{7}$$

where n is an arbitrary positive integer. A detonation front would then coincide with the line $\eta = \eta_D =$ const. The detonation front velocity is given by

$$U = \frac{dr}{dt} = - n C_1 \eta_D^{1/n} r^{(1-(1/n))}. \tag{8}$$

Consequently the initial conditions are functions of η_D and r only

$$p_2 = \frac{\gamma}{\gamma + 1} n^2 \eta_D^{2/n} r^{2(1-(1/n))} p_1 \tag{9a}$$

$$\rho_2 = \frac{\gamma + 1}{\gamma} \rho_1 \tag{9b}$$

$$v_2 = - \frac{n}{\gamma + 1} \eta_D^{1/n} r^{(1-(1/n))} C_1. \tag{9c}$$

Solution

The following transformations are made for the pressure, density and flow velocity distributions.

$$p = \rho_1 r^{2(1-(1/n))} P(\eta) \tag{10a}$$

$$\rho = \rho_1 R(\eta) \tag{10b}$$

$$v = r^{1-(1/n)} V(\eta) \tag{10c}$$

Introducing Eqns. (10a, 10b, 10c) into Eqns. (2) – (4), we get the following set of ordinary differential equations.

$$(n C_1 \eta^{1+(1/n)} + V\eta) \frac{dR}{d\eta} + R\eta \frac{dV}{d\eta} + (1 - (1/n) - k + \sigma) RV = 0 \tag{11}$$

$$(n C_1 \eta^{1+(1/n)} + V\eta) \frac{dV}{d\eta} + \frac{\eta}{R} \frac{dP}{d\eta} + (1 - (1/n)) V^2 + (2 - (2/n) - k) \frac{P}{R} = 0 \tag{12}$$

$$(n C_1 \eta^{1+(1/n)} + V\eta) \left(\frac{dP}{d\eta} - \gamma \frac{P}{R} \frac{dR}{d\eta} \right) + [k(\gamma - 1) + 2(1 - (1/n))] PV = 0 \tag{13}$$

Using a modified sound speed A, given by $A^2 = \gamma \frac{P}{R}$, we get the following two differential equations for A and V.

$$\frac{1}{A} \frac{dA}{d\eta} = \frac{-\frac{\gamma - 1}{2} \eta \frac{dV}{d\eta} - \frac{1}{2}[(\gamma - 1)\sigma + (\gamma + 1)(1 - (1/n))] V}{(n C_1 \eta^{1+(1/n)} + V\eta} \tag{14}$$

$$\frac{dV}{d\eta} = \frac{1}{\gamma [(n C_1 \eta^{1+(1/n)} + V\eta)^2 - \eta^2 A^2]} \{A^2 V \eta [\gamma(1 - (1/n) - k + \sigma) + k(\gamma - 1) + 2(1 - (1/n))] - [(1 - (1/n))(\gamma V^2 + 2A^2) - kA^2] \times (n C_1 \eta^{1+(1/n)} + V\eta)\}. \tag{15}$$

with the initial conditions,

$$A = \frac{\gamma}{\gamma + 1} n \eta_D^{1/n} C_1 \tag{16a}$$

and

$$V = -\frac{n}{\gamma + 1} \eta_D^{1/n} C_1. \tag{16b}$$

It can be easily seen that the denominator of Eqn. (15) vanishes for the initial values. Thus the numerator of Eqn. (15) has to vanish too for a finite derivative of V which gives us a formula for the determination of n

$$n = \frac{3(\gamma + 1)}{(3 - k)(\gamma + 1) + \sigma\gamma} \tag{17}$$

For singularity considerations we follow the same way as Teipel[5] did. The variable η is eliminated from Eqns. (14) and (15) by using

$$\mu = \frac{A}{n C_1 \eta^{1/n}} \tag{18a}$$

and

$$\nu = -\frac{V}{n C_1 \eta^{1/n}}. \tag{18b}$$

Then remains now only one differential equation.

$$\frac{d\mu}{d\nu} = \frac{\mu}{2(\nu - 1)} \cdot \frac{1}{\mu^2[\nu(\sigma + 1)\gamma + 2(1 - (1/n)) - k] - \gamma\nu(\nu - 1)(\nu - (1/n))}$$
$$\times \{2\mu^2[\gamma(\nu - 1) + 1 - (1/n)] - k\mu^2(\gamma - 1) - \gamma(\nu - 1)\langle (2 - \sigma$$
$$+ \sigma\gamma)\nu^2 - \gamma\left[\sigma(\gamma - 1) + \gamma + 1 + \frac{3 - \gamma}{n}\right] + (2/n)\rangle\}, \tag{19}$$

with the initial conditions

$$\mu_D = \frac{\gamma}{\gamma + 1} \tag{20a}$$

and

$$\nu_D = \frac{1}{\gamma + 1}. \tag{20b}$$

There are several singular points of Eqn. (19), which can be seen from its integration. Because we are interested in the detonation waves running towards the centre, the integral curve has to pass through initial conditions $\left(\mu_D = \frac{\gamma}{\gamma + 1}, \nu_D = \frac{1}{\gamma + 1}\right)$ and centre ($x = 0, t = 0$) which is the correct curve for the given problem and represents the solution.

Pressure Distribution

Evaluating Eqn. (9a) in combination with Eqn. (17), we can calculate the pressure jump for the detonation front. For $\gamma = 1.4$ and $k = 0.5$, we obtain

$$\sigma = 1, \frac{p_2}{p_1} \sim r^{-0.055} \qquad (r^{-0.388})$$

$$\sigma = 2, \frac{p_2}{p_1} \sim r^{-0.444} \qquad (r^{-0.777})$$

The values in the brackets relate to the case of uniform gas at rest. We assume that at a certain initial radius $r = r_0$ and at a certain time $t = t_0$ the detonation front has initiated its own motion by the Chapman-Jouguet velocity U_0. Hence from Eqns. (8) and (9a) the pressure distribution can be given as

$$\frac{p_2}{p_1}\frac{C_1^2}{U_0^2} = \frac{\gamma}{\gamma+1}\left(\frac{r}{r_0}\right)^{2(1-(1/n))}$$

Conclusions

Taking the initial front velocity $U_0 = 2C_1$, the pressure distribution has been shown in Fig. 1. It is observed that there is a remarkable change in the pressure distribution on account of varying density. A similar result is obtained for the temperature

Table 1. Values of n for different values k at $\gamma = 1.4$

k	$\sigma = 1$	$\sigma = 2$
0	0.8372	0.72
0.5	0.9729	0.8182
1.0	1.1613	0.9473
1.5	1.44	1.125
2.0	1.8947	1.3846
2.5	2.7692	1.8
3.0	5.1428	2.5714

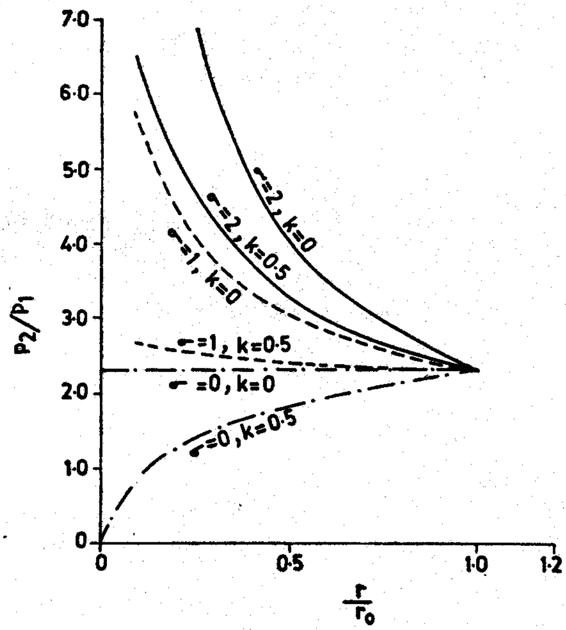

Figure 1. Pressure distribution.

distribution. The detonation with spherical symmetry is much stronger in the neighbourhood of the centre than the cylindrical wave. The values of n for different values of k are shown in Table 1 in the cases $\sigma = 1$ and $\sigma = 2$. The range of k depends on γ and for $\gamma = 1.4$ it is in the range, $0 \leqslant k < 3.58$ for $\sigma = 1$ and $0 \leqslant k < 4.16$ for $\sigma = 2$. It is found that n increases considerably due to the increase in k. The increase, however, is less for $\sigma = 2$ than that for $\sigma = 1$.

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