Def Sci J, Vol 32, No 4, October 1982, pp. 303–312

Rocket Rendezvous at Preassigned Destinations with Optimum Exit Trajectories

T. N. Srivastava

Institute for Systems Studies & Analysis, Old Secretariat, Delhi-110 054

Ashok Kumar Nangia

U.P.S,C. Computer Centre, New Delhi-110 011

Received 30 March 1981

Abstract. The problem of rendezvous of an interceptor rocket vehicle through optimal exit path with a destination rocket vehicle at a preassigned location on the destination orbit has been investigated for non-coaxial coplanar elliptic launch and destination orbits in an inverse square gravitational field. The case, when launch and destination orbits are coplanar circular, is also discussed. In the end numerical results for rendezvous have been obtained taking Earth and Mars orbit as launch and destination orbits respectively.

Nomenclature

θ = Vectorial angle

u = Reciprocal of radius vector

E = Eccentricity

h = Twice the aerial velocity

K = Gravitational parameter

α = Angle of inclination of major axis of destination orbit with reference line, that is, the line joining the force centre (focus) and the pericentre of the launch orbit

β = Angle of inclination of major axis of transfer trajectory with reference line

V = Velocity

γ = Heading angle, that is, angle between velocity vector and local horizontal

a = Length of semi-major axis

Subscripts

l = Corresponds to launch orbit

d = Corresponds to destination orbit

i = Corresponds to velocity-injection point

1 = Relates to parameters of transfer trajectory before velocity injection point

2 = Relates to parameters of transfer trajectory after velocity injection point

e = Denotes values just after impulsive velocity change of interceptor rocket at launch point

f = Denotes values just before impulsive velocity change of interceptor rocket at velocity injection point

g = Denotes values just after impulsive velocity change of interceptor rocket at velocity injection point

Superscripts

$(-)$ = relates to optimum transfer trajectory

1. Introduction

Paiewonsky[1] has treated the problem of rocket rendezvous for a very restricted case of coplanar concentric circular orbits taking only Hohmann transfer trajectories. Later, for non-coaxial coplanar elliptic orbits, the problem of interception of a rocket vehicle by optimal path at a preassigned point on the destination orbit by a commuter rocket vehicle launched from the pericentre of the launch orbit was solved by evaluating launch angle and orbital parameters of the transfer trajectory[2]. In this context two assumptions have been made. Firstly, that no intermediate velocity injection is given to the commuter rocket vehicle and secondly, that the commuter rocket vehicle is launched from the pericentre of the launch orbit. Both the assumptions have been relaxed in the present paper which analyses the problem of rendezvous at a preassigned location by an interceptor rocket vehicle with a rocket on the destination orbit through optimal exit path. The interceptor rocket vehicle after having been launched from the launch orbit is imparted a velocity impulse at a specified intermediate space point to achieve rendezvous. The optimum exit path is characterized by the minimum fuel expenditure required in launching and imparting intermediate velocity injection to the interceptor rocket vehicle to achieve rendezvous with the destination rocket.

Elements of the optimum exit path to achieve rendezvous and launch angle, that is, the vectorial angle of the destination rocket vehicle at the instant launch rocket vehicle is fired so as to achieve rendezvous with the former at the preassigned space point on the destination orbit have been investigated. Analysis for the case when both launch and destination orbits are coplanar circular has also been done. In the end, numerical results for the rendezvous have been obtained taking Earth and Mars orbit as launch and destination orbit respectively.

2. Flight Durations of Rocket Vehicles and Launch Angle

Let the equations of the launch and destination orbits be

$$h_l^2 u = K[1 + E_l \cos\theta] \tag{1}$$

$$h_d^2 u = K[1 + E_d \cos(\theta - \alpha)] \tag{2}$$

Also let the equations of the transfer trajectory of interceptor rocket before and after the application of intermediate velocity impulse be respectively given by

$$h_1^2 u = K[1 + E_1 \cos(\theta - \beta_1)] \tag{3}$$

$$h_2^2 u = K[1 + E_2 \cos(\theta - \beta_2)] \tag{4}$$

By conservation of momentum principle, the flight time T_1 of interceptor rocket from the launch point (u_l, θ_l) to the velocity injection point (u_i, θ_i) is given by

$$T_1 = \int_0^{T_1} dt = \frac{h_1^3}{K^2} \int_{\theta_l}^{\theta_i} \frac{d\theta}{[1 + E_1 \cos(\theta - \beta_1)]^2}$$

$$= \frac{a_1^{3/2}}{\sqrt{K}} \Bigg[2\tan^{-1}\left\{\left(\frac{1 - E_1}{1 + E_1}\right)^{1/2} \tan\left(\frac{\theta_i - \beta_1}{2}\right)\right\}$$

$$- 2\tan^{-1}\left\{\left(\frac{1 - E_1}{1 + E_1}\right)^{1/2} \tan\left(\frac{\theta_l - \beta_1}{2}\right)\right\}$$

$$- E_1(1 - E_1^2)^{1/2}\left\{\frac{\sin(\theta_i - \beta_1)}{1 + E_1\cos(\theta_i - \beta_1)} - \frac{\sin(\theta_l - \beta_1)}{1 + E_1\cos(\theta_l - \beta_1)}\right\}\Bigg] \tag{5}$$

The time of flight $(T_2 - T_1)$ of the intercepter rocket from the velocity injection point to the destination point (u_d, θ_d) on the destination orbit will be

$$T_2 - T_1 = \int_{T_1}^{T_2} dt = \frac{h_2^3}{K^2} \int_{\theta_i}^{\theta_d} \frac{d\theta}{[1 + E_2 \cos(\theta - \beta_2)]^2}$$

$$= \frac{(a_2)^{3/2}}{\sqrt{K}} \Bigg[2\tan^{-1}\left\{\left(\frac{1 - E_2}{1 + E_2}\right)^{1/2} \tan\left(\frac{\theta_d - \beta_2}{2}\right)\right\}$$

$$- 2\tan^{-1}\left\{\left(\frac{1 - E_2}{1 + E_2}\right)^{1/2} \tan\left(\frac{\theta_i - \beta_2}{2}\right)\right\}$$

$$- E_2(1 - E_2^2)^{1/2}\left\{\frac{\sin(\theta_d - \beta_2)}{1 + E_2\cos(\theta_d - \beta_2)} - \frac{\sin(\theta_i - \beta_2)}{1 + E_2\cos(\theta_i - \beta_2)}\right\}\Bigg] \tag{6}$$

Where T_2 is the total time of flight of the interceptor rocket. For flight time τ of the destination rocket a procedure similar as above gives

$$\tau = \int_0^T dt = \frac{h_d^3}{K^2} \int_{\theta_0}^{\theta_d} \frac{d\theta}{[1 + E_d \cos(\theta - \alpha)]^2}$$

$$= \frac{(a_d)^{3/2}}{\sqrt{K}} \left[2 \tan^{-1} \left\{ \left(\frac{1 - E_d}{1 + E_d} \right)^{1/2} \tan \left(\frac{\theta_d - \alpha}{2} \right) \right\} \right.$$

$$- 2 \tan^{-1} \left\{ \left(\frac{1 - E_d}{1 + E_d} \right)^{1/2} \tan \left(\frac{\theta_0 - \alpha}{2} \right) \right\}$$

$$\left. - E_d (1 - E_d^2)^{1/2} \left\{ \frac{\sin(\theta_d - \alpha)}{1 + E_d \cos(\theta_d - \alpha)} - \frac{\sin(\theta_0 - \alpha)}{1 + E_d \cos(\theta_0 - \alpha)} \right\} \right] \quad (7)$$

Where (u_0, θ_0) is the position of the destination rocket at the instant interceptor rocket is launched. For rendezvous of the interceptor rocket with destination rocket at the preassigned destination point (u_d, θ_d), the following condition should be satisfied

$$\tau = T_2 = (T_2 - T_1) + T_1 \quad (8)$$

Where T_1, $(T_2 - T_1)$ and τ are given by (5), (6) and (7) respectively. Eqn. (8) gives launch θ_0 for any value of θ_d, if the elements of the transfer trajectory (E_1, E_2, a_1, a_2, β_1 and β_2) are known.

3. Elements of Optimum Transfer Trajectory for Rendezvous

The characteristic velocity ΔV of the interceptor rocket can be written as

$$\Delta V = |\Delta V_1| + |\Delta V_2| = |[V_i^2 + V_e^2 - 2V_i V_e \cos(\gamma_e - \gamma_i)]^{1/2}|$$

$$+ |[V_f^2 + V_g^2 - 2V_f V_g \cos(\gamma_g - \gamma_f)]^{1/2}| \quad (9)$$

By principle of conservation of angular momentum we have

$$V_f u_t \cos \gamma_f = V_e u_i \cos \gamma_e \quad (10)$$

Also V_e and V_f can be related as

$$V_f^2 - V_e^2 = 2K(u_i - u_t) \quad (11)$$

Substitution of (10) and (11) in (9) gives

$$\Delta V = |[V_i^2 + V_e^2 - 2V_i V_e \cos(\gamma_e - \gamma_i)]^{1/2}|$$

$$+ |[V_g^2 - 2V_g [V_e d_1 \cos \gamma_e \cos \gamma_g + \sin \gamma_g \{V_e^2 + 2K(u_i - u_t)$$

$$-V_e^2\, d_1^2 \cos^2 \gamma_e\}^{1/2}] + V_e^2 + 2K\,(u_e - u_l)]^{1/2}\,| \tag{12}$$

Where

$$d_1 = u_l/u_t$$

The relationship between V_e and γ_e can be expressed[3] as

$$V_e^2 = \frac{K u_t \{1 - \cos(\theta_e - \theta_l)\} \sec^2 \gamma_e}{\{d_1 + \sin(\theta_e - \theta_l) \tan \gamma_e - \cos(\theta_e - \theta_l)\}} \tag{13}$$

Similarly we can write

$$V_g^2 = \frac{K u_t \{1 - \cos(\theta_d - \theta_l)\} \sec^2 \gamma_g}{\{d_2 + \sin(\theta_d - \theta_l) \tan \gamma_g - \cos(\theta_d - \theta_l)\}} \tag{14}$$

Where

$$d_2 = u_d/u_t$$

For minimum ΔV we have

$$\frac{\partial(\Delta V)}{\partial \gamma_e} = 0 \quad , \quad \frac{\partial(\Delta V)}{\partial \gamma_g} = 0 \tag{15}$$

Substitution from Eqns. (12), (13) and (14) in Eqn. (15) and differentiation gives

$$\frac{1}{\Delta V_1}\left[\lambda - 2V_l\left\{\mu \cos(\gamma_e - \gamma_l) - V_e \sin(\gamma_e - \gamma_l)\right\}\right]$$
$$+ \frac{1}{\Delta V_2}\left[\lambda - 2V_g\left[d_1 \cos \gamma_g\left\{\mu \cos \gamma_e - V_e \sin \gamma_l\right\}\right.\right.$$
$$\left.\left. + \frac{\sin \gamma_g}{2\nu}\left\{\lambda\,(1 - d_1^2 \cos^2 \gamma_e + 2V_e^2\, d_1^2 \cos \gamma_e \sin \gamma_e\right\}\right]\right] = 0 \tag{16}$$

and

$$[V_g - \{V_e\, d_1 \cos \gamma_e \cos \gamma_g + \nu \sin \gamma_g\}]$$
$$\times\left[\tan \gamma_g - \frac{\sin(\theta_d - \theta_l) \sec^2 \gamma_g}{2\{d_2 + \sin(\theta_d - \theta_l) \tan \gamma_g - \cos(\theta_d + \theta_l)\}}\right]$$
$$+ V_e\, d_1 \cos \gamma_e \sin \gamma_g - \nu \cos \gamma_g = 0 \tag{17}$$

where

$$\lambda = 2\left[\frac{K u_t \{1 - \cos(\theta_e - \theta_l)\} \sec^2 \gamma_e}{d_1 + \sin(\theta_e - \theta_l) \tan \gamma_e - \cos(\theta_e - \theta_l)}\right]^{1/2} \mu$$

$$= \frac{K u_t \{1 - \cos(\theta_e - \theta_l)\} \sec^2 \gamma_e}{(d_1 + \sin(\theta_e - \theta_l) \tan \gamma_e - \cos(\theta_e - \theta_l))^2}$$

$$\times\, [2 \tan \gamma_e\,(d_1 - \cos(\theta_e - \theta_l) + \sin(\theta_e - \theta_l)(\tan^2 \gamma_e - 1)]$$

$$v = \left[V_e^2 + 2K(u_t - u_i) - V_e^2 d_1^2 \cos^2 \gamma_e \right]^{1/2}$$

Eqns. (16) and (17) can be solved numerically for γ_e and γ_g giving optimum departure angles $\bar{\gamma}_e$ and $\bar{\gamma}_g$ which when substituted in (13) and (14) will give optimum departure velocities $\bar{V}_e$ and $\bar{V}_g$ respectively.

Having obtained $\bar{\gamma}_e$, $\bar{\gamma}_g$, $\bar{V}_e$, $\bar{V}_g$ elements of the optimum transfer trajectory $\bar{E}_1$, $\bar{E}_2$, $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{a}_1$ and $\bar{a}_2$ can be obtained by following relationships of space dynamics[3]

$$\bar{E}_1 = \left[\left(\frac{\bar{V}_e^2}{u_i K} - 1 \right) \cos^2 \bar{\gamma}_e + \sin^2 \bar{\gamma}_e \right]^{1/2} \tag{18}$$

$$\bar{E}_2 = \left[\left(\frac{\bar{V}_g^2}{u_i K} - 1 \right) \cos^2 \bar{\gamma}_g + \sin^2 \bar{\gamma}_g \right]^{1/2} \tag{19}$$

$$\bar{\beta}_1 = \theta_i - \tan^{-1}\left(\frac{\bar{V}_e^2 \sin \bar{\gamma}_e \cos \bar{\gamma}_e}{\bar{V}_e^2 \cos^2 \bar{\gamma}_e - K u_i} \right) \tag{20}$$

$$\bar{\beta}_2 = \theta_i - \tan^{-1}\left(\frac{\bar{V}_g^2 \sin \bar{\gamma}_g \cos \bar{\gamma}_g}{\bar{V}_g^2 \cos^2 \bar{\gamma}_g - K u_i} \right) \tag{21}$$

$$\bar{a}_1 = \frac{K}{2Ku_i - \bar{V}_e^2} \tag{22}$$

$$\bar{a}_2 = \frac{K}{2Ku_i - \bar{V}_g^2} \tag{23}$$

Substituting the values of the elements of optimum transfer trajectory given by (18–23) in (8), the optimum launch angle $\bar{\theta}_0$ for rendezvous for interception angle θ_d can be obtained.

4. Rendezvous Between Circular Launch and Destination Orbits

For circular launch and destination orbits

$$E_i = E_d = \alpha = 0 \tag{24}$$

Substituting from (24) in (7), we have

$$\tau = \frac{(a_d)^{3/2}}{\sqrt{K}} (\theta_d - \theta_0) \tag{25}$$

Where a_d in (25) now signifies orbital radius of the destination orbit. Substituting from (25) in (8) yields

$$\theta_0 = \theta_d - \frac{\sqrt{K}}{(a_d)^{3/2}}\left[(T_2 - T_1) + T_1\right] \tag{26}$$

Eqn. (26) gives the launch angle θ_0 for rendezvous between circular launch and destination orbits where T_1 and $(T_2 - T_1)$ are given by Eqns. (5) and (6) respectively. Expression for T_1 given by Eqn. (5) can be simplified in this case by taking the reference line as the line joining the force centre and launch point thus making $\theta_l = 0$.

In equations giving $\bar{\gamma}_e$ and $\bar{\gamma}_\theta$, viz Eqns. (16) and (17), now

$$\gamma_l = 0 \text{ and } d_1 = a_l u_l, \;\; d_2 = (a_d u_l)^{-1}$$

where a_l is the orbit radius of the launch orbit. Also V_l in (16) now stands for the circular orbital velocity corresponding to the launch orbit.

Elements of the optimum trajectory $\bar{E}_2$, $\bar{\beta}_2$, $\bar{a}_2$ will be given as before by Eqns. (19), (21) and (23) respectively but $\bar{E}_1$, $\bar{\beta}_1$ and $\bar{a}_1$ are now given by

$$\bar{E}_1 = \left[\left(\frac{a_l \bar{V}_e^2}{K} - 1\right)\cos^2\bar{\gamma}_e + \sin^2\bar{\gamma}_e\right]^{1/2}$$

$$\bar{\beta}_1 = \tan^{-1}\left[\frac{a_l \bar{V}_e^2 \sin\bar{\gamma}_e \cos\bar{\gamma}_e}{K - a_l(\bar{V}_e \cos\bar{\gamma}_e)^2}\right]$$

$$\bar{a}_1 = \frac{a_l K}{2K - a_l \bar{V}_e^2}.$$

5. Rendezvous Between Earth and Mars

Launch angles and elements of the corresponding optimum transfer trajectories have been calculated as a numerical illustration for rendezvous at preassigned destinations between Earth and Mars. Earth and Mars orbits are taken as circular and coplanar and their orbital radii as (1.49×10^8) km and (2.28×10^8) km respectively. It is assumed that intermediate velocity injection is applied to the interceptor rocket vehicle where $\theta_i = \theta_d/2$ and it is moving at a distance of 1.25 a.u from the sun, the force centre.

Table 1 gives the launch angles and elements of the transfer trajectory before velocity injection for optimum rendezvous between Earth and Mars for different interception angles. Elements of the transfer trajectory after the velocity injection for optimum rendezvous are given in Table 2.

Variation of the launch angle and elements of optimum transfer trajectory for rendezvous between Earth and Mars with respect to interceptor angle are shown in Figs. 1–4. A study of the Figs. brings out the following interesting results:

(i) Variation of launch angle is slow for lower values of interception angle and gradually becomes fast with increase of interception angle (Fig. 1).

(ii) Variation of lengths of semi-major axis of optimum transfer trajectory before and after velocity injection ($\bar{a}_1$ and $\bar{a}_2$) is fast for lower values of interception

Table 1. Launch angles and elements of transfer trajectory before velocity injection ($\bar{E}_1$, $\bar{\beta}_1$ and $\bar{a}_1$) for optimum rendezvous between Earth and Mars

Interception angle θ_d	Launch angle θ_0	Eccentricity $\bar{E}_1$	Inclination angle of the major axis $\bar{\beta}_1$	Semi-major axis times 10^{-8} ($\bar{a}_1 10^{-8}$) km
180°	100° 14′	0.4458	349° 5′	1.858
150°	80° 55′	0.4475	332° 2′	1.850
120°	62° 55′	0.4542	313° 43′	1.837
90°	46° 0′	0.4695	292° 50′	1.808
60°	38° 35′	0.5213	269° 27′	1.764
45°	33° 50′	0.5623	252° 6′	1.660

Table 2. Elements of the transfer trajectory after velocity injection ($\bar{E}_2$, $\bar{\beta}_2$ and $\bar{a}_2$) for optimum rendezvous between Earth and Mars

Interception angle θ_d	Eccentricity $\bar{E}_2$	Inclination angle of the major axis $\bar{\beta}_2$	Semi-major axis times 10^{-8} ($\bar{a}_2 10^{-8}$) km
180°	0.2262	171° 1′	1.881
150°	0.2149	152° 43′	1.863
120°	0.2120	132° 31′	1.835
90°	0.2441	113° 08′	1.814
60°	0.3332	90° 46′	1.778
45°	0.3879	69° 59′	1.650

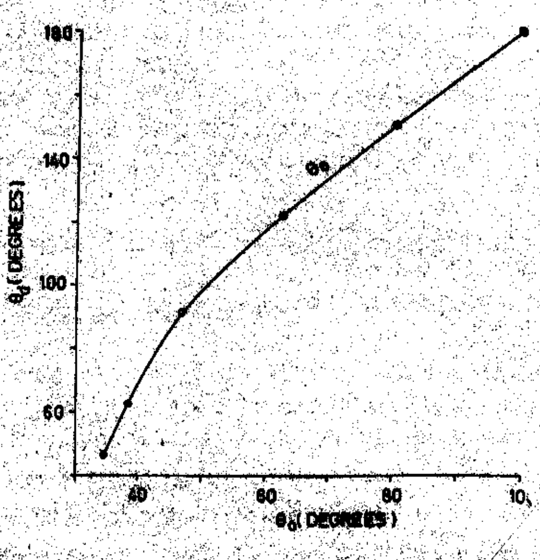

Figure 1. Variation of launch angle θ_0 with respect to interception on angle θ_d.

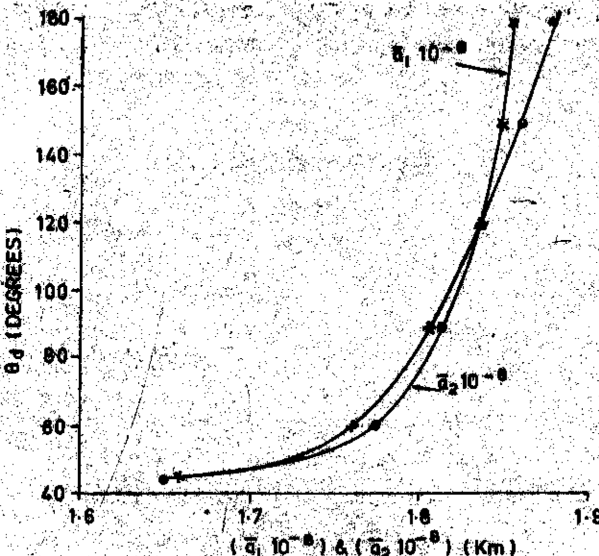

Figure 2. Variation of lengths of semi-major axes $\bar{a}_1$ and $\bar{a}_2$ with respect to interception angle θ_d.

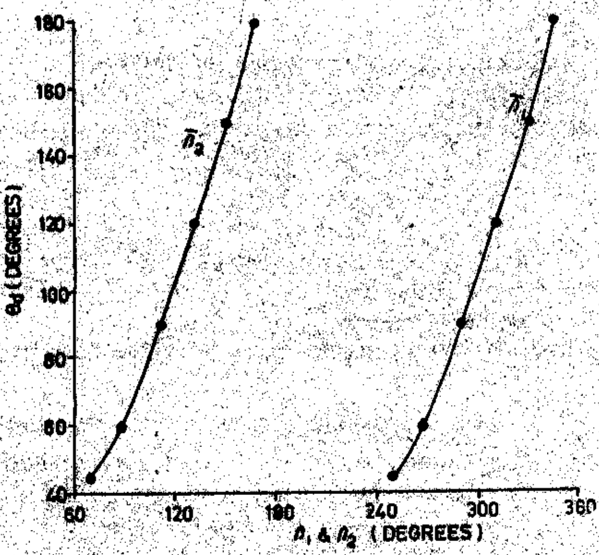

Figure 3. Variation of angles of inclination $\bar{\beta}_1$ and $\bar{\beta}_2$ with respect to interception angle θ_d.

angle but gradually becomes slow with increasing value of interceptor angle (Fig. 2).

(iii) Angles of inclination of major axis of optimum transfer trajectory before and after velocity injection ($\bar{\beta}_1$ and $\bar{\beta}_2$) vary almost uniformly with respect to interception angle (Fig. 3).

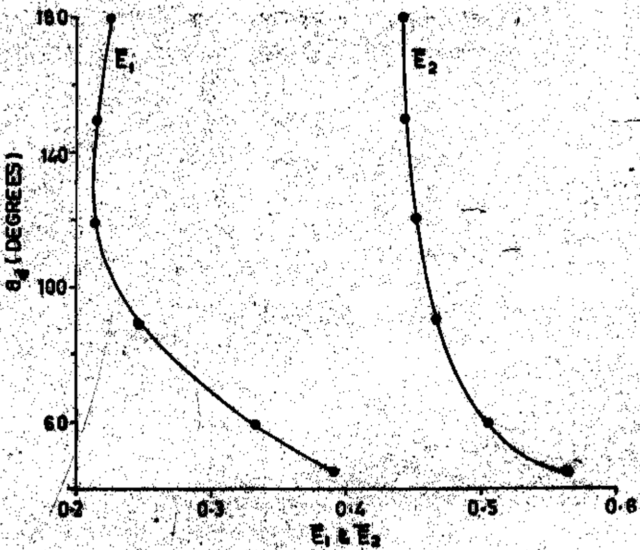

Figure 4. Variation of eccentricities $\bar{E_1}$ and $\bar{E_2}$ with respect to interception angle θ_4.

Acknowledgements

Authors are thankful to Maj. Gen. A. Balasubramanian, AVSM, Director, EDP Systems, Delhi and Dr L. K. Wadhwa, Director, Institute for Systems Studies & Analysis, Delhi for their encouragements in the preparation of this paper.

References

1. Paiewonsky, B. H., *Jet Propulsion*, 28 (1958), 121.
2. Srivastava, T. N., *Def. Sci. J.*, 18 (1968), 147–160.
3. Thomson, W. T., 'Introduction to Space Dynamics' (John Wiley and Sons, Inc., New York, London).