OPTIMIZATION PROBLEM OF MULTISTAGE ROCKET

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The necessary conditions for the existence of minimum of a function of initial and final values of mass, position and velocity components and time of a multistage rocket have been reviewed when the thrust levels in each stage are considered to be bounded and variation in gravity with height has been taken into account. The nature of the extremal subarcs comprising the complete extremal arc has been studied. A few simple examples have been given as illustrations.

There have been various studies on the optimization process for a single stage vehicle[1-5]. In a recent paper Leitmann[6] solved the general problem of extremising a function termed as the "payoff" or "performance index", and dependent upon the initial and final values of mass, position and velocity components and time of a single stage rocket, and derived the necessary conditions for the existence of such extrema and the nature of the extremal arc. The problem was treated as a variational problem of the Bolza-Mayer type with limitations imposed on controls. Recently the author[7] applied the variational calculus to solve the general problem of extremising a given "payoff" for a multiple n-stage rocket moving in vacuum. In the present paper the problem of multistage rocket has been generalised to include the case when the engines of various stages are capable of delivering all mass flows between a lower limit and an upper limit, i. e., the thrust of various stages are considered to be limited and variation into gravity with height is accounted for. The necessary conditions for the existence of a minimum "payoff" and the nature of extremal subarcs in various stages comprising the complete extremal arc have been studied. A few simple illustrations have been given.

EQUATIONS OF MOTION AND FORMULATION OF THE PROBLEM

The motion of a n-stage rocket is governed by the following differential equations.

$$\left.\begin{aligned}
\dot{x} &= u && (1)\\
\dot{h} &= v && (2)\\
\dot{u} &= \frac{C_i\,\beta_i}{m_i}\cos\theta_i && (3)\\
\dot{v} &= \frac{C_i\,\beta_i}{m_i}\sin\theta_i - g_0\left(1-\frac{2h}{r}\right) && (4)\\
\dot{m}_i + \beta_i &= 0 && (5)
\end{aligned}\right\}\quad i = 1, 2, \ldots, n$$

where x denotes a horizontal coordinate; h a vertical coordinate; u and v the components of velocity, m the mass; θ the inclination of flight path with respect to the horizon; g_0 the acceleration due to gravity at sea level; r the radius of earth; C the equivalent exit velocity of a rocket engine and β is the mass flow. The dot sign denotes a derivative with respect to time. Since the mass flow rate is assumed to be between two limits, i. e.

$$\overset{*}{\beta_i} \leqslant \beta_i \leqslant \overset{**}{\beta_i} \quad i = 1, 2, \ldots, n$$

we can replace it by the equality constraint.

$$(\beta_i - \overset{*}{\beta_i})(\overset{**}{\beta_i} - \beta_i) - \alpha_i^2 = 0 \quad i = 1, 2, \ldots, n \tag{6}$$

where α_i is a real variable.

The system of equations (1)—(6) contains eight physical variables x, h, u, v, m_i, β_i, θ_i and α_i for any particular stage and there are six equations connecting them. Thus there are two variables say β_i and θ_i for any particular stage which can be controlled and hence are called control variables, the other five quantities x, h, u, v, and m_i being termed as the state variables. Therefore the variational problem is formulated as follows: In the class of functions $x(t)$, $h(t)$, $u(t)$, $v(t)$, $m_i(t)$, $\beta_i(t)$ and $\alpha_i(t)$ which are consistent with equations (1)—(6) and certain prescribed end conditions, it is required to find that particular set which minimises a certain function of the form

$$J \equiv J\,[\,x(t_0), h(t_0), v(t_0), m(t_0), t_0, x(t_f), h(t_f), v(t_f), m(t_f), t_f] \tag{7}$$

Let us now combine the equations (1) to (6) to form the following auxiliary function for the nth stage.

$$F_i \equiv \lambda x_i(\dot{x} - u) + \lambda_{hi}(\dot{h} - v) + \lambda_{ui}\left(\dot{u} - \frac{C_i\,\beta_i}{m_i}\cos\theta_i\right) + \lambda_{vi}\left[\dot{v} + g_0\left(1 - \frac{2h}{r}\right) - \frac{C_i\,\beta_i}{m_i}\sin\theta_i\right] + \lambda_{mi}(\dot{m}_i + \beta_i) + \lambda_{\beta i}\left[(\beta_i - \overset{*}{\beta}_i)(\overset{**}{\beta}_i - \beta_i) - \alpha_i^2\right] \tag{8}$$

$$i = 1, 2, \ldots, n$$

where λ_{xi}, λ_{hi}, λ_{ui}, λ_{vi} λ_{mi} and $\lambda_{\beta i}$ are undetermined Lagrangian multipliers and are functions of time since the constraining equations must also be satisfied at all points of the trajectory.

EULER-LAGRANGE EQUATIONS

A necessary condition for J to be minimum is that the extremal path must not only satisfy equations (1) to (6) but also the Euler-Lagrange equations given by

$$\frac{d}{dt}\left(\frac{F_i}{\partial Z_j}\right) = \frac{\partial F_i}{\partial Z_j} \qquad i = 1, 2, \ldots, n \tag{9}$$

where Z_j are the problem variables of the stage under consideration, i. e. $x, h, u, v, m, \theta, \beta$ and α. The Euler-Lagrange equations for the given stage for the present problem may therefore be written explicitly as

$$\dot{\lambda}_{xi} = 0 \tag{10}$$

$$\dot{\lambda}_{hi} = -\lambda_{vi}\frac{2}{r}g_0 \tag{11}$$

$$\dot{\lambda}_{ui} = -\lambda_{xi} \tag{12}$$

$$\dot{\lambda}_{vi} = -\lambda_{hi} \tag{13}$$

$$\dot{\lambda}_{mi} = \lambda_{ui}\frac{C_i\,\beta_i}{m_i}\cos\theta_i + \lambda_{vi}\frac{C_i\,\beta_i}{m_i}\sin\theta_i \tag{14}$$

$$0 = \frac{C_i\,\beta_i}{m_i}\left(\lambda_{ui}\sin\theta_i - \lambda_{vi}\cos\theta_i\right) \tag{15}$$

$$0 = -\lambda_{ui}\frac{C_i}{m_i}\cos\theta_i - \lambda_{vi}\frac{C_i}{m_i}\sin\theta_i + \lambda_{mi} + \lambda_{\beta i}(\overset{**}{\beta}_i + \overset{*}{\beta}_i - \alpha\beta_i) \tag{16}$$

$$\lambda_{\beta i}\,\alpha_i = 0 \tag{17}$$

These equations and the subsequent equations will hold good for each stage separately and therefore for the sake of convenience the subscript i will be dropped. From equation (15)

$$\tan\theta = \frac{\lambda_u}{\lambda_v} \tag{18}$$

FIRST INTEGRAL

Since the function F does not contain the independent variable t explicitly, a first integral of the Euler-Lagrange equations can be readily written with the help of

$$\left(F - \sum_{j=1}^{\infty} \frac{\partial F}{\partial \dot{Z}_j} \dot{Z}_j \right) = C'$$

i. e.

$$C' + \lambda_x u + \lambda_h v + \lambda_u \frac{C\beta}{m} \cos\theta + \lambda_v \left[\frac{C\beta}{m} \sin\theta - g_0 \left(1 - \frac{2h}{r} \right) \right] - \lambda_m \beta = 0 \qquad (19)$$

where C' is an integration constant having different values in different stages.

TRANSVERSALITY CONDITION

Transversality condition of the problem, which gives changes in boundary conditions as also change in J must be specified and is given by

$$dJ + \Big[\lambda_x dx + \lambda_h dh + \lambda_u du + \lambda_v dv + \lambda_m dm - C' dt \Big]_{t_f}^{t_i} = 0 \qquad (20)$$

From here we obtain the following boundary conditions of the problem

$$\left.\begin{aligned} \lambda_x(t_0) &= -\frac{\partial J}{\partial x(t_0)}, & \lambda_h(t_0) &= -\frac{\partial J}{\partial h(t_0)} \\ \lambda_u(t_0) &= -\frac{\partial J}{\partial u(t_0)}, & \lambda_v(t_0) &= -\frac{\partial J}{\partial v(t_0)} \\ \lambda_m(t_0) &= -\frac{\partial J}{\partial m(t_0)}, & C'(t_0) &= \frac{\partial J}{\partial t_0} \end{aligned}\right\} \quad \begin{aligned} \lambda_x(t_f) &= \frac{\partial J}{\partial x(t_f)}, & \lambda_h(t_f) &= \frac{\partial J}{\partial h(t_f)} \\ \lambda_u(t_f) &= \frac{\partial J}{\partial u(t_f)}, & \lambda_v(t_f) &= \frac{\partial J}{\partial v(t_f)} \\ \lambda_m(t_f) &= \frac{\partial J}{\partial m(t_f)}, & C'(t_f) &= -\frac{\partial J}{\partial t_f} \end{aligned} \left.\vphantom{\begin{aligned}a\\a\\a\end{aligned}}\right\} \qquad (21)$$

CORNER CONDITIONS

The boundary conditions on the Lagrangian multipliers at staging point or at any point where the thrust becomes discontinuous are derived from the corner conditions, i. e. $\partial F / \partial \dot{Z}_j$ and $\left(-F + \Sigma \frac{\partial F}{\partial \dot{Z}_j} \dot{Z}_j \right)$ must be continuous at such points. For the present problem if t_i's $\{i = 1, 2, \ldots, (n-1)\}$ are the instants where staging takes place, then

$$\begin{aligned} \lambda_{xi}(t_i) &= \lambda_{xi+1}(t_i) \\ \lambda_{hi}(t_i) &= \lambda_{hi+1}(t_i) \\ \lambda_{ui}(t_i) &= \lambda_{ui+1}(t_i) \\ \lambda_{vi}(t_i) &= \lambda_{vi+1}(t_i) \\ \lambda_{mi}(t_i) &= \lambda_{mi+1}(t_i) \end{aligned}$$

and

$$\begin{aligned} &\left[\lambda_{xi} u + \lambda_{hi} v - \lambda_{vi} g_0 \left(1 - \frac{2h}{t} \right) - \lambda_{mi} \beta_i + \frac{C_i \beta_i}{m_i} \left(\lambda_{ui} \cos\theta_i + \lambda_{vi} \sin\theta_i \right) \right]_{t_i} \\ &= \Bigg[\lambda_{xi+1} u + \lambda_{hi+1} v - \lambda_{vi+1} g_0 \left(1 - \frac{2h}{r} \right) - \lambda_{mi+1} \beta_{i+1} + \\ &\quad + \frac{C_{i+1} \beta_{i+1}}{m_{i+1}} \left(\lambda_{ui+1} \cos\theta_{i+1} + \lambda_{vi+1} \sin\theta_{i+1} \right) \Bigg]_{t_i} \\ &i = 1, 2, \ldots, (n-1) \end{aligned} \qquad (22)$$

Thus the Lagrangian multipliers λ_x, λ_h, λ_u, λ_v, λ_m are continuous across staging and hence continuous throughout the period of flight. Therefore from (18) we see that there is no discontinuity in the angle θ at the staging points.

WEIERSTRASS CONDITION

A necessary condition for the minimum value of J is

$$E \geqslant 0$$

where

$$E = F(\overset{\dagger}{Z_j}, \overset{\dagger}{\dot{Z}_j}) - F(Z_j, \dot{Z}_j) - \sum_{j=1}^{8} \left(\overset{\dagger}{\dot{Z}_j} - \dot{Z}_j \right) \frac{\partial F}{\partial \dot{Z}_j}$$

Here † denotes functions subjected to finite admissible variations. This on evaluation gives

$$K_{\beta_i} \beta_i \geqslant \overset{\dagger}{K}_{\beta_i} \overset{\dagger}{\beta_i} \qquad i = 1, 2, \ldots, n \tag{23}$$

where

$$\left.\begin{aligned} K_{\beta_i} &= \frac{C_i}{m_i} \left(\lambda_u \cos\theta + \lambda_v \sin\theta \right) - \lambda_m \\ \overset{\dagger}{K}_{\beta_i} &= \frac{C_i}{m_i} \left(\lambda_u \cos\overset{\dagger}{\theta} + \lambda_v \sin\overset{\dagger}{\theta} \right) - \lambda_m \end{aligned}\right\} i = 1, 2, \ldots, n \tag{24}$$

Equation (23) must be satisfied for all admissible variations, not all zero.

BURNING PROGRAMME

Differentiating equation (24) with respect to time and simplifying by making use of Euler-Lagrange equations, we obtain.

$$\dot{K}_{\beta_i} = - \frac{C_i}{m_i} \left(\lambda_x \cos\theta + \lambda_h \sin\theta \right) \tag{25}$$

From (17) we observe that the extremum arc is discontinuous and we can have subarcs of the following type:

either $\quad (i)\ \lambda_\beta \neq 0, \quad \alpha = 0$

or $\quad (ii)\ \lambda_\beta = 0, \quad \alpha \neq 0$

The first possibility implies that either $\beta = \beta^*$ or $\beta = \beta^{**}$ while the second possibility means that $\dot{K}_\beta = 0$, i.e. $K_\beta = 0$ which is incompatible with (25) and accordingly the first possibility is forced meaning thereby that only subarcs of minimum or maximum thrust can exist in the various stages and there is no subarc flown with intermediate thrust in any one of the stages.

SEQUENCE OF SUBARCS

Since the possibility of intermediate thrust subarc is ruled out, we have now to determine the conditions for the existence of minimum and maximum thrust subarcs. Now whenever $\beta = \beta^*$ or $\beta = \beta^{**}$, the condition (23) must hold all along the extremal arc, which means that when $K_\beta = \overset{\dagger}{K}_\beta$, then

$$\left.\begin{aligned} K_\beta < 0 \qquad \beta = \beta^* \\ K_\beta > 0 \qquad \beta = \beta^{**} \end{aligned}\right\} \tag{26}$$

But when $\quad K_\beta \neq \overset{\dagger}{K}_\beta$ and $\beta = \overset{\dagger}{\beta}$ then

$$K_\beta \geqslant \overset{\dagger}{K}_\beta$$

i. e.

$$\lambda_u \cos\theta + \lambda_v \sin\theta \geqslant \lambda_u \cos\overset{\dagger}{\theta} + \lambda_v \sin\overset{\dagger}{\theta} \tag{27}$$

implying that for all admissible variations in θ, the value of θ should be such that $(\lambda_u \cos\theta + \lambda_v \sin\theta)$ is always maximum. Thus we conclude that when K_β changes sign then there is a change in the nature of the thrust programming in any given stage and the arc is of maximum thrust if $K_\beta > 0$ and of minimum thrust if $K_\beta < 0$. Hence K_β is the quantity whose behaviour determines the nature of the extremal arc in the various stages.

Now as a consequence of the last equation of (22), we find that

$$(K_{\beta_i}\,\beta_i)_{t_i} = (K_{\beta_{i+1}}\,\beta_{i+1})_{t_i} \tag{28}$$

Since β_i's are always positive, this shows that at the staging point K_{β_i} and $K_{\beta_{i+1}}$ are of the same sign and hence there is no change in the characteristics of the thrust in the transfer from one stage to the other, i. e. the next stage takes over with the same kind of thrust as that of the final point of the previous stage.

ILLUSTRATIONS

We will illustrate the above by taking a few simple examples of vertically ascending two stage rocket. The constraining equations in this case are given by

$$\left.\begin{aligned} \dot v + g_0\left(1 - \frac{2h}{r}\right) - \frac{C_i\,\beta_i}{m_i} &= 0 \\ \dot h &= v \\ \dot m_i + \beta_i &= 0 \\ (\beta_i - \overset{*}{\beta_i})\,(\overset{**}{\beta_i} - \beta_i) - \alpha_i^2 &= 0 \end{aligned}\right\}_{i=1,2} \tag{29}$$

The auxiliary function is given by

$$F \equiv \lambda_v\left[\dot v + g_0\left(1 - \frac{2h}{r}\right) - \frac{C_i\,\beta_i}{m_i}\right] + \lambda_h\left(\dot h - v\right) + \lambda_m\left(\dot m_i + \beta_i\right) +$$
$$+ \lambda_\beta\left[(\beta_i - \overset{*}{\beta_i})\,(\overset{**}{\beta_i} - \beta_i) - \alpha_i^2\right] = 0 \tag{30}$$

The Euler-Lagrange equations are

$$\dot\lambda_v + \lambda_h = 0 \tag{31}$$

$$\dot\lambda_h = -\frac{2g_0}{r}\,\lambda_v \tag{32}$$

$$\dot\lambda_m = \frac{C_i\,\beta_i}{m_i^2}\,\lambda_v \tag{33}$$

$$-\frac{C_i}{m_i}\,\lambda_v + \lambda_\beta\left(\overset{**}{\beta_i} - \overset{*}{\beta_i} - 2\,\beta_i\right) + \lambda_m = 0 \tag{34}$$

$$\lambda_\beta\,\alpha_i = 0 \tag{35}$$

The transversality condition is

$$dJ + \Big[\lambda_v\,d v + \lambda_h\,d h + \lambda_m\,d m_i - C_i\,d t\Big]_{t_0}^{t_2} = 0 \tag{36}$$

The first integral is

$$\lambda_v\,\dot v + \lambda_h\,\dot h + \lambda_m\,\dot m_1 = C_i \tag{37}$$

From (35), we see that either $\lambda_\beta = 0$ or $\alpha_i = 0$. $\lambda_\beta = 0$ implies subarc flown with variable thrust while $\alpha_i = 0$ means that the thrust is either maximum or minimum.

Now (31) and (32) may be integrated to give

$$\left.\begin{aligned} \lambda_v(t) &= \lambda_v(t_2)\cosh\omega(t_2-t) + \frac{\lambda_h(t_2)}{\omega}\sinh\omega(t_2-t) \\ \lambda_h(t) &= \omega\,\lambda_v(t_2)\sinh\omega(t_2-t) + \lambda_h(t_2)\cosh\omega(t_2-t) \end{aligned}\right\} \tag{38}$$

where
$$\omega = \frac{2g_0}{r}$$

Now in order to draw some conclusions it will be advisable to take gravity as constant throughout the flight period and in that case the set of equations (38) will reduce to

$$\left.\begin{aligned} \lambda_h(t) &= e \\ \lambda_v(t) &= \lambda_v(t_2) + e\,(t_2-t) \end{aligned}\right\} \tag{39}$$

where e is an integration constant.

In the following examples we will take the minimum thrust subarc to be coasting subarc, i. e. $\beta^* \to 0$.

Example 1—Maximum Final Height.

In this case $J \equiv -h(t_2)$

The boundary conditions of the given problem are

$$t = t_0 = 0 \begin{cases} h = h_0 \\ v = v_0 \\ m = m_0 \end{cases} \qquad t = t_2 \text{ (unspecified)} \begin{cases} v = v_2 \\ \\ m = m_2 \end{cases}$$

Since the final time is not prescribed and with the aid of prescribed boundary conditions, the transversality condition (36) leads to

$$C(t_2) = 0\,, \quad \lambda_h(t_2) = 0$$

In this case therefore $e = 1$

Also here

$$\dot{K}_i = -\frac{C_i}{m_i} \qquad i = 1, 2 \tag{40}$$

This shows that K_1 and K_2 both have only one zero, i. e. in each stage there can only be one corner point and hence almost two subarcs in each stage. If we integrate (40), we have

$$K_i = \frac{C_i}{\beta_i^{**}} \log \frac{m}{m_{C_i}} \qquad \beta_i = \beta_i^{**} \tag{41}$$

$$K_i = \frac{C_i}{m_{C_i}}\left(t_{C_i} - t\right) \qquad \beta_i = 0 \tag{42}$$

where subscripts C_i $(i = 1, 2)$ refer to the instant of the corner point. Now if $\beta_i = \beta_i^{**}$ then obviously from (41) $K_i > 0$ and if $\beta_i = 0$ then from (42) $K_i \leqslant 0$. Also at the final point ($t = t_2$) only (42) can hold since $t_2 > t_{C_2}$; i. e. the final point is reached with a coasting subarc. Again since in each stage there can be only one corner point, we see that for the first stage there can be two possibilities

$$\left.\begin{array}{llll} \text{either} & \text{(i)} & \beta_1 = \beta_1^{**} & t_0 \leqslant t \leqslant t_1 \\ \text{or} & \text{(ii)} & \beta_1 = \beta_1^{**} & t_0 \leqslant t \leqslant t_{C_1} \\ & & \beta_1 = 0 & t_{C_1} \leqslant t \leqslant t_1 \end{array}\right\} \tag{43}$$

But since

$$\left(K_{\beta_1}\beta_1\right)_{t_1} = \left(K_{\beta_2}\beta_2\right)_{t_1}$$

we observe that the following corresponding subarcs are possible in the second stage

$$\text{(i)}\ \beta_2 = \beta_2^{**}, \quad t_1 \leqslant t \leqslant t_{c_2}, \quad \beta_2 = 0, \quad t_{c_2} \leqslant t \leqslant t_2 \quad \text{(ii)}\ \beta_2 = 0, \quad t_1 \leqslant t \leqslant t_2$$

Fig. 1 illustrates the possible optimum controls associated with the problem of achieving maximum final height.

Example 2—Maximum Final Velocity.

In this case $J = -v\,(t_2)$

The prescribed boundary conditions are

$$t = t_0 = 0 \begin{cases} h = h_0 \\ v = v_0 \\ m = m_0 \end{cases} \qquad t = t_2 \text{ (unspecified) } \{m = m_2\}$$

Since the final time is not prescribed and making use of the given boundary conditions the transversality condition (36) gives

$$C\,(t_2) = 0\,, \qquad \lambda_h\,(t_2) = 0, \qquad \lambda_v\,(t_2) = 1 \tag{44}$$

Therefore (39) implies that $e = 0$

Also here $$\dot{K}_{\beta_i} = 0 \quad (i = 1, 2)$$

i. e. K_{β_i} is constant throughout the flight.

Also making use of the first integral (37), we obtain at the final point

$$\beta_2\,(t_2)\,K_{\beta_2}\,(t_2) = g, \text{ i. e. } K_{\beta_2}\,(t_2) > 0$$

and hence the final point is reached with a maximum thrust subarc.

Now since K_β has no zero in a given stage, the optimum subarcs to achieve the given mission must be of the form

$$\left.\begin{aligned} \beta_1 &= \beta_1^{**} \qquad && 0 \leqslant t \leqslant t_1 \\ \beta_2 &= \beta_2^{**} \qquad && t_1 \leqslant t \leqslant t_2 \end{aligned}\right\} \tag{45}$$

The optimum control associated with the problem of maximum final velocity is indicated in Fig. 2.

Example 3—Maximum Payload or Minimum Fuel Consumption.

In this case $J \equiv -m\,(t_2)$

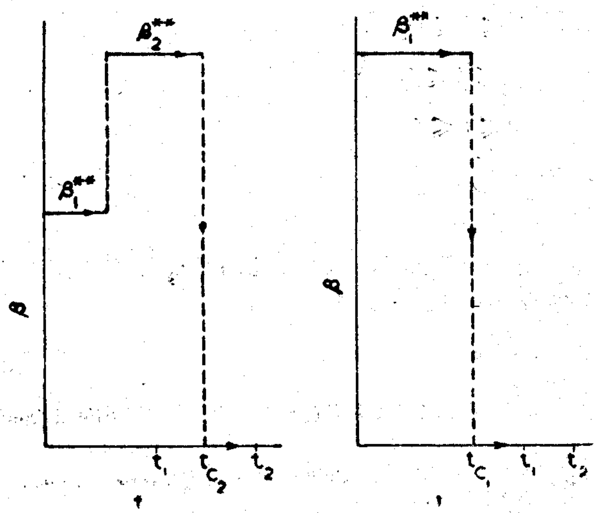

Fig. 1—Mass flow rate versus time for a two-stage rocket for achieving maximum height.

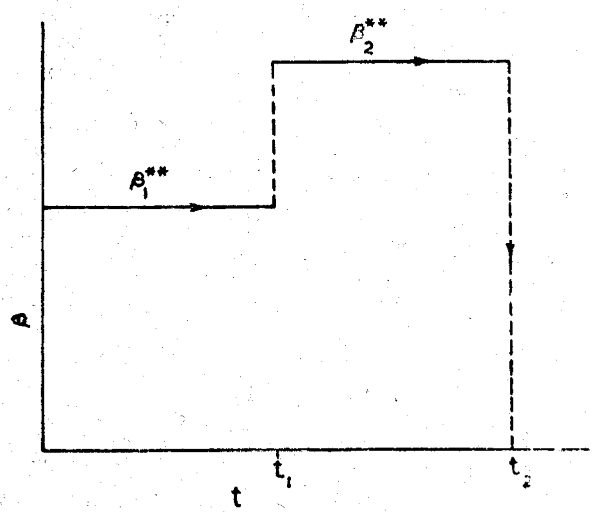

Fig. 2—Mass flow rate versus time for a two-stage rocket for achieving maximum velocity.

The boundary conditions are specified by

$$t = t_0 = 0 \begin{cases} h = h_0 \\ v = v_0 \\ m = m_0 \end{cases} \qquad t = t_2 \text{ (unspecified) } \{ h = h_2 \}$$

Since the final time is not specified and with the help of the given boundary conditions we obtain

$$C(t_2) = 0 , \qquad \lambda_m(t_2) = 1 , \qquad \lambda_v(t_2) = 0 \tag{46}$$

Also in this case

$$\lambda_h = e , \qquad \lambda_v = e(t_2 - t) , \qquad K_{\beta_i} = \frac{C_i}{m_i}\lambda_v - \lambda_m \tag{47}$$

Therefore

$$\dot{K}_{\beta_i} = \frac{C_i}{m_i} e t_2$$

Thus K_{β_i} does not change sign during the stage and hence cannot be zero more than once in any stage. Also at the final point

$$K_{\beta_2}(t_2) = -1$$

Therefore at the final point K_{β_2} is negative, i. e. $\beta_2(t_2) = 0$ and hence the final arc is a coasting subarc.

Now making use of the first integral we obtain

$$v(t_2) = \frac{\beta_2(t_2)}{e} \tag{48}$$

But as already observed that final point is attained with a coasting subarc, therefore (48) implies that the velocity is zero at the final point. Now we make use of the relation

$$(K_{\beta_1}\beta_1)_{t_1} = (K_{\beta_2}\beta_2)_{t_1}$$

We have already seen that K_{β_1} and K_{β_2} cannot have more than one zero each and therefore for the first stage there can be two possible modes of propulsions, i. e.

$$\left.\begin{array}{llll} \text{either} & \text{(i)} & \beta_1 = \beta_1^{**} & t_0 \leqslant t \leqslant t_1 \\ \text{or} & \text{(ii)} & \beta_1 = \beta_1^{**} & t_0 \leqslant t \leqslant t_{c_1} \\ & & \beta_1 = 0 & t_{c_1} \leqslant t \leqslant t_1 \end{array}\right\} \tag{49}$$

The corresponding modes of propulsion in the second stage will be

$$\left.\begin{array}{lll} \text{(i)} & \beta_2 = \beta_2^{**} & t_1 \leqslant t \leqslant t_{c_2} \\ & \beta_2 = 0 & t_{c_2} \leqslant t \leqslant t_2 \\ \text{(ii)} & \beta_2 = 0 & t_1 \leqslant t \leqslant t_2 \end{array}\right\} \tag{50}$$

Thus the possible modes of operation to achieve the given mission are the same as illustrated in Fig. 1.

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