SOME THEOREMS ON GENERALIZED HYPERGEOMETRIC FUNCTION AND KAMPÉ DE FÉRIET FUNCTION

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Two theorems on the sums of generalized hypergeometric functions have been established. The theorems have further been employed to prove some theorems on the sums of Kampé de Fériet functions.

Two theorems on the sums of generalized hypergeometric functions have been established. In the first theorem we have expressed a ${}_{p+1}F_{q+1}$, in which a numerator parameter exceeds a denominator parameter by a positive integer, say m, as the sum of $m+1$ ${}_pF_q$. In the second theorem we have expressed a ${}_{p+1}F_{q+1}$ in which a denominator parameter exceeds a numerator parameter by a positive integer say n, as the sum of $n+1$ ${}_{p+1}F_{q+1}$. The two theorems have further been used to establish some theorems on the sums of Kampé de Fériet functions. On specializing the parameters, the theorems yield many results some of which are known while the others are believed to be new.

The hypergeometric function of two variables defined by Kampé de Fériet[1] is denoted as :

$$F\begin{matrix} g:h\,;H \\ \\ i:j\,;J \end{matrix}\left[\begin{matrix} a_g:b_h\,;B_H\,; \\ \\ \alpha_i:c_j\,;C_J\,; \end{matrix}\; x, y\right] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{\Pi(a_g)_{p+q}\,\Pi(b_h)_p\,\Pi(B_H)_q\,x^p y^q}{\Pi(\alpha_i)_{p+q}\,\Pi(c_j)_p\,\Pi(C_J)_q\,p!q!} \tag{1}$$

where μ_i denotes the set of parameters $\mu_1,\ldots,\mu_i$. $\Pi(\mu_i)_s$ denotes the product $(\mu_1)_s\ldots(\mu_i)_s$ $(\mu)_s$, denotes $\Gamma(\mu+s)\,/\,\Gamma(\mu)$. Colon (:) and semicolon (;) separate the terms of the form $(\alpha_i)_{p+q}$ and $(c_j)_p$, $(C_J)_q$.

Ragab[2] has shown that the series on the right of (1) converges for all finite x, y if $g+h \leqslant i+J$, $g+H \leqslant i+J$, converges for $|x| + |y| < \min(1, 2^{i-g+1})$ if $g+h = i+j+1$, $g+H = i+J+1$. It can also be shown that the series converges if

$$g+h = i+j+1, \quad g+H = i+J+1$$

and $\qquad |x| \leqslant 1, \quad |y| < 1, \quad \mathrm{Re}(X) > 0,$

or $\qquad |x| < 1, \quad |y| \leqslant 1, \quad \mathrm{Re}(Y) > 0,$

or $\qquad |x| \leqslant 1, \quad |y| \leqslant 1, \quad \mathrm{Re}(X) > 0,\ \mathrm{Re}(Y) > 0,$

where

$$X = \Sigma\alpha_i + \Sigma c_j - \Sigma a_g - \Sigma b_h$$

$$Y = \Sigma\alpha_i + \Sigma C_J - \Sigma a_g - \Sigma B_H$$

The following result will be required in the proof[3].

$$\begin{aligned}(a-c+1)\ {}_{p+1}F_{q+1}(a_p\,,\ a\,;\ b_q\,,\ c\,;x) \\ = a\ {}_{p+1}F_{q+1}(a_p\,,\ a+1\,;\ b_q\,,\ c\,;\ x) - (c-1)\ {}_{p+1}F_{q+1}(a_p\,,a\,;b_q\,,c-1\,;x)\end{aligned} \tag{2}$$

THEOREMS ON THE SUMS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

The theorems to be established are

$${}_{p+1}F_{q+1}(a_p, a+n\,; b_q, a\,; x) = \sum_{i=0}^{n}\binom{n}{i}\frac{\Pi(a_p)_i\,x^i}{\Pi(b_q)_i\,(a)_i}\,{}_pF_q(a_p+i\,; b_q+i\,; x) \tag{3}$$

where n is a positive integer.

$$ {}_{p+1}F_{q+1}(a_p, a-n\,;\, b_q, a\,;\, x) = \sum_{i=0}^{n} \binom{n}{i} \frac{\Pi(a_p)_i\,(-x)^i}{\Pi(b_q)_i\,(a)_i} \times {}_{p+1}F_{q+1}(a_p+i, a;\, b_q+i, a+i;\, x) \quad (4) $$

where n is a positive integer.

Proof :

We shall prove the above theorems by induction on n.

We have

$$ {}_{p+1}F_{q+1}(a_p, a+1\,;\, b_q, a;\, x) = \sum_{r=0}^{\infty} \frac{\Pi(a_p)_r\, x^r}{\Pi(b_q)_r\, r!} \left(1 + \frac{r}{a}\right) $$

$$ = {}_pF_q(a_p\,;\, b_q\,;\, x) + \frac{\Pi(a_p)\, x}{\Pi(b_q)\, a}\, {}_pF_q(a_p+1\,;\, b_q+1\,;\, x) $$

Thus (3) holds for $n = 1$.

Let us suppose that it holds for $n = m$. Further

$$ {}_{p+1}F_{q+1}(a_p, a+m+1\,;\, b_q, a\,;\, x) = \sum_{r=0}^{\infty} \frac{\Pi(a_p)_r\, x^r}{\Pi(b_q)_r\, r!} \frac{(a+m)_r}{(a)_r} \left[1 + \frac{r}{a+m}\right] $$

$$ = {}_{p+1}F_{q+1}(a_p, a+m\,;\, b_q, a\,;\, x) + \frac{\Pi(a_p)x}{\Pi(b_q)a} $$

$$ {}_{p+1}F_{q+1}(a_p+1,\, a+m+1\,;\, b_q+1,\, a+1\,;\, x) $$

Now using the result (3) for $n = m$ and the elementary results

$$ \binom{m}{r} + \binom{m}{r-1} = \binom{m+1}{r}, \binom{m}{0} = \binom{m+1}{0}, \binom{m}{m} = \binom{m+1}{m+1} $$

we find that the theorem holds for $m+1$. This proves (3).

We see that (4) holds for $n = 1$. Let us assume that it holds for $n = m$. Further (2) yields

$$ {}_{p+1}F_{q+1}(a_p, a-m-1\,;\, b_q, a\,;\, x) - {}_{p+1}F_{q+1}(a_p, a-m\,;\, b_q, a\,;\, x) $$

$$ = -\frac{a-1}{m}\left[{}_{p+1}F_{q+1}(a_p, a-m\,;\, b_q, a\,;\, x) - {}_{p+1}F_{q+1}(a_p, a-m-1\,;\, b_q, a-1\,;\, x)\right] \quad (5) $$

Using the assumption that (4) holds for $n = m$, we see that the right hand side of (5), on using the summation notation for F, becomes

$$ -\frac{a-1}{m} \sum_{i=0}^{m} \sum_{r=0}^{\infty} \frac{\Pi(a_p)_{i+r}\,(-1)^i\, x^{i+r}}{\Pi(b_q)_{i+r}\; r!} \binom{m}{i} \left[\frac{(a)_r}{(a)_{i+r}} - \frac{(a-1)_r}{(a-1)_{i+r}}\right] $$

$$ = -\frac{a-1}{m} \sum_{i=0}^{m} \sum_{r=0}^{\infty} \binom{m}{i} \frac{\Pi(a_p)_{i+r}\,(-)^i\, x^{i+r}\,(-i)\,(a-1)_r}{\Pi(b_q)_{i+r}\; r!\; (a-1)_{r+i+1}} $$

$$ = \sum_{i=1}^{m} \sum_{r=0}^{\infty} \binom{m-1}{i-1} \frac{\Pi(a_p)_{i+r}\,(-)^i\, x^{i+r}\,(a)_r}{\Pi(b_q)_{i+r}\; r!\; (a)_{i+r}} \left[1 - \frac{r}{a+r-1}\right] $$

$$ = \sum_{i=0}^{m} \binom{m-1}{i-1} \frac{\Pi(a_p)_i\,(-x)^i}{\Pi(b_q)_i\,(a)_i}\; {}_{p+1}F_{q+1}(a_p+i, a\,;\, b_q, a+i\,;\, x) - $$

$$ -\sum_{i=2}^{m+1} \binom{m-1}{i-2} \frac{\Pi(a_p)_{i+r-1}\,(-)^{i-1}\, x^{i+r-1}\,(a)_{r-1}}{\Pi(b_q)_{i+r-1}\,(a)_{i+r-1}\,(r-1)!} $$

Here we have changed i to $i-1$ in the second expression. By virtue of

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

We have

$$\sum_{i=1}^{m+1} \frac{\Pi(a_p)_i\,(-x)^i}{\Pi(b_q)_i\,(a)_i}\; {}_{p+1}F_{q+1}\,(a_p+i\,,\,a\,;\,b_q+i\,,\,a+i\,;x)$$

Substituting in (5) and again using

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

we see that the theorem (4) is established for $n = m+1$.

THEOREMS ON THE SUMS OF KAMPÉ DE FÉRIET FUNCTIONS

Theorems to be established are

$$F\begin{matrix} g:h;H+1 \\ \\ i:j;J+1 \end{matrix}\left[\begin{matrix} a_g:b_h;B_H\,,\,B+n; \\ & x,\,y \\ \alpha_i:c_j;C_J,\,B; \end{matrix}\right]$$

$$= \sum_{\lambda=0}^{n} \binom{n}{\lambda} \frac{\Pi(a_g)_\lambda\,\Pi(B_H)_\lambda\,y^\lambda}{\Pi(\alpha_i)_\lambda\,\Pi(C_J)_\lambda\,(B)_\lambda}\; F\begin{matrix} g:h;H \\ \\ i:j;J \end{matrix}\left[\begin{matrix} a_g+\lambda:b_h;B_H+\lambda; \\ & x,\,y \\ \alpha_i+\lambda:c_j;\,C_J+\lambda; \end{matrix}\right] \tag{6}$$

$$F\begin{matrix} g:h+1;H+1 \\ \\ i:j+1;J+1 \end{matrix}\left[\begin{matrix} a_g:b_h\,,\,b+m;\,B_H\,,\,B+n; \\ & x,\,y \\ \alpha_i:\,c_j\,,\,b; \quad C_J\,,\,B; \end{matrix}\right]$$

$$= \sum_{\lambda=0}^{m}\sum_{\mu=0}^{n} \binom{m}{\lambda}\binom{n}{\mu} \frac{\Pi(a_g)_{\lambda+\mu}\,\Pi(b_h)_\lambda\,\Pi(B_H)_\mu\,x^\lambda\,y^\mu}{\Pi(\alpha_i)_{\lambda+\mu}\,\Pi(c_j)_\lambda\,\Pi(C_J)_\mu\,(b)_\lambda\,(B)_\mu}\,.$$

$$.\,F\begin{matrix} g:h;H \\ \\ i:j;J \end{matrix}\left[\begin{matrix} a_g+\lambda+\mu:b_h+\lambda;\,B_H+\mu; \\ & x,\,y \\ \alpha_i+\lambda+\mu:c_j+\lambda;\,C_J+\mu; \end{matrix}\right] \tag{7}$$

$$F\begin{matrix} g+1:h;H \\ \\ i+1:j;J \end{matrix}\left[\begin{matrix} a_g\,,\,a+n:b_h;\;B_H; \\ & x,\,y \\ \alpha_i\,,\quad a:\;c_j;\quad C_J; \end{matrix}\right]$$

$$= \sum_{\lambda=0}^{n}\sum_{\mu=0}^{n-\lambda} \binom{n}{\lambda}\binom{n-\lambda}{\mu} \frac{\Pi(a_g)_{\lambda+\mu}\;\Pi(b_h)_\lambda\;\Pi(B_H)_\mu\;x^\lambda\,y^\mu}{\Pi(\alpha_i)_{\lambda+\mu}\;\Pi(c_j)_\lambda\;\Pi(C_J)_\mu\;(a)_{\lambda+\mu}}\,.$$

$$.\,F\begin{matrix} g:h;H \\ \\ i:j;J \end{matrix}\left[\begin{matrix} a_g+\lambda+\mu:b_h+\lambda;\,B_H+\mu; \\ & x,\,y \\ \alpha_i+\lambda+\mu:c_j+\lambda;\,C_J+\mu; \end{matrix}\right] \tag{8}$$

$$F\begin{matrix} g:h;H+1 \\ \\ i:j;J+1 \end{matrix}\left[\begin{matrix} a_g:b_h;\,B_H\,,\,B-n; \\ & x,\,y \\ \alpha_i:c_j;\;C_J\,,\;B; \end{matrix}\right]$$

$$= \sum_{\lambda=0}^{n} \binom{n}{\lambda} \frac{\Pi(a_g)_\lambda\,\Pi(B_H)_\lambda\,(-y)^\lambda}{\Pi(\alpha_i)_\lambda\,\Pi(C_J)_\lambda\,(B)_\lambda}\; F\begin{matrix} g:h;H+1 \\ \\ i:j;J+1 \end{matrix}\left[\begin{matrix} a_g+\lambda:\,b_h;\,B_H+\lambda,\,B; \\ & x,\,y \\ \alpha_i+\lambda:\,c_j;\;C_J+\lambda,\,B+\lambda; \end{matrix}\right] \tag{9}$$

$$F\,{}^{g:h+1;H+1}_{i:j+1;J+1}\left[\begin{matrix} a_g : b_h\,,\, b-m\,;\, B_H\,,\, B-n\,; \\ \alpha_i : c_j\,,\, b\,; \qquad C_J\,,\, B\,; \end{matrix}\; x, y\right]$$

$$= \sum_{\lambda=0}^{m} \sum_{\mu=0}^{n} \binom{m}{\lambda} \binom{n}{\mu} \frac{\Pi(a_g)_{\lambda+\mu}\; \Pi(b_h)_\lambda\; \Pi(B_H)_\mu\; (-x)^\lambda\; (-y)^\mu}{\Pi(\alpha_i)_{\lambda+\mu}\; \Pi(c_j)_\lambda\; \Pi(C_J)_\mu\; (b)_\lambda\; (B)_\mu}\,.$$

$$.\; F\,{}^{g:h+1;H+1}_{i:j+1;J+1}\left[\begin{matrix} a_g+\lambda+\mu : b_h+\lambda, b\,; \qquad B_H+\mu\,,\, B\,; \\ \alpha_i+\lambda+\mu : c_j+\lambda,\, b+\lambda\,;\, C_J+\mu\,,\, B+\mu\,; \end{matrix}\; x, y\right] \tag{10}$$

$$F\,{}^{g:h+1;H+1}_{i:j+1;J+1}\left[\begin{matrix} a_g : b_h\,,\, b-m\,;\, B_H\,,\, B+n\,; \\ \alpha_i : c_j\,,\, b\,; \qquad C_J\,,\, B\,; \end{matrix}\; x, y\right]$$

$$= \sum_{\lambda=0}^{m} \sum_{\mu=0}^{n} \binom{m}{\lambda} \binom{n}{\mu} \frac{\Pi(a_g)_{\lambda+\mu}\; \Pi(b_h)_\lambda\; \Pi(B_H)_\mu\; (-x)^\lambda\; (y)^\mu}{\Pi(\alpha_i)_{\lambda+\mu}\; \Pi(c_j)_\lambda\; \Pi(C_J)_\mu\; (b)_\lambda\; (B)_\mu}\,.$$

$$.\; F\,{}^{g:h+1;H}_{i:j+1;J}\left[\begin{matrix} a_g+\lambda+\mu : b_h+\lambda\,,\, b\,; \qquad B_H+\mu\,; \\ \alpha_i+\lambda+\mu : c_j+\lambda\,,\, b+\lambda\,;\, C_J+\mu\,; \end{matrix}\; x, y\right] \tag{11}$$

$$F\,{}^{g+1:h;H}_{i+1:j;J}\left[\begin{matrix} a_g\,,\, a-n : b_h\,;\, B_H\,; \\ \alpha_i\,,\, a : \qquad c_j\,;\; C_J\,; \end{matrix}\; x, y\right]$$

$$= \sum_{\lambda=0}^{n} \sum_{\mu=0}^{n} \binom{n}{\lambda} \binom{n}{\mu} \frac{\Pi(a_g)_{\lambda+\mu}\; (-x)^\mu\; (-y)^\lambda\; (B_H)_\lambda\; (b_h)_\mu}{\Pi(\alpha_i)_{\lambda+\mu}\; (C_J)_\lambda\; (c_j)_\mu\; (a)_{\lambda+\mu}}\,.$$

$$.\; F\,{}^{g+1:h+1;H}_{i+1:j+1;J}\left[\begin{matrix} a_g+\lambda+\mu,\; a+\mu : a, \qquad b_h+\mu\,;\, B_H+\lambda\,; \\ \alpha_i+\lambda+\mu\,;\, a+\lambda+\mu : a+\mu\,,\, c_j+\mu\,;\, C_J+\lambda\,; \end{matrix}\; x, y\right] \tag{12}$$

$$F\,{}^{g+2:h;H}_{i+2:j;J}\left[\begin{matrix} a_g\,,\, a+m\,,\, A-n : b_h\,;\, B_H\,; \\ \alpha_i\,,\;\; a\,,\;\; A\,:\, c_j\,; \qquad C_J\,; \end{matrix}\; x, y\right]$$

$$= \sum_{\rho=0}^{n} \sum_{\nu=0}^{n} \sum_{\lambda=0}^{m} \sum_{\mu=0}^{m-\lambda} \binom{n}{\rho} \binom{n}{\nu} \binom{m}{\lambda} \binom{m-\lambda}{\mu} \frac{\Pi(a_g)_{\lambda+\mu+\rho+\nu}\, \Pi(A-n)_{\lambda+\mu}\, \Pi(b_h)_{\lambda+\nu}\, \Pi(B_H)_{\mu+\rho}}{\Pi(\alpha_i)_{\lambda+\mu+\rho+\nu}\, \Pi(A)_{\lambda+\mu+\nu+\rho}\, \Pi(c_j)_{\lambda+\nu}\, \Pi(C_J)_{\mu+\rho}}$$

$$.\; \frac{x^{\lambda+\nu}\; y^{\mu+\rho}\; (-)^{\rho+\nu}}{(a)_{\lambda+\mu}}\,.$$

$$.\; F\,{}^{g+1:h+1;H}_{i+1:j+1;J}\left\{\begin{matrix} a_g+\lambda+\mu+\rho+\nu,\; A+\lambda+\mu+\rho : A+\lambda+\mu; \\ b_h+\lambda+\nu\,;\, B_H+\rho+\mu\,; \\ \alpha_i+\lambda+\mu+\rho+\nu,\; A+\lambda+\mu+\rho+\nu : \\ A+\lambda+\mu+\rho\,,\, c_j+\lambda+\nu\,;\, C_J+\mu+\rho\,; \end{matrix}\; x, y\right\} \tag{13}$$

Proof :

We have

$$F = F\begin{matrix} g:h;H+1 \\ \\ i:j;J+1\end{matrix}\left[\begin{matrix} a_g : b_h ; B_H , B+n ; \\ & x, y \\ \alpha_i : c_j ; C_J , B ; \end{matrix}\right]$$

$$= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\Pi(a_g)_{p+q}\, \Pi(b_h)_p\, \Pi(B_H)_q\, (B+n)_q\, x^p\, y^q}{\Pi(\alpha_i)_{p+q}\, \Pi(c_j)_p\, \Pi(C_J)_q\, (B)_q\, p!\, q!}$$

$$= \sum_{p=0}^{\infty} \frac{\Pi(a_g)_p\, \Pi(b_h)_p\, x^p}{\Pi(\alpha_i)_p\, \Pi(c_j)_p\, p!}\ {}_{g+H+1}F_{i+J+1}\,(a_g+p\,,B_H\,,B+n;\alpha_i+p\,,C_J\,,B\,;y)$$

Using (3) , we get

$$F = \sum_{p=0}^{\infty} \frac{\Pi(a_g)_p\, \Pi(b_h)_p\, x^p}{\Pi(\alpha_i)_p\, \Pi(c_j)_p\, p!} \sum_{\lambda=0}^{n} \binom{n}{\lambda} \frac{\Pi(a_g+p)_\lambda\, (B_H)_\lambda\, y^\lambda}{\Pi(\alpha_i+p)_\lambda\, (C_J)_\lambda\, (B)_\lambda}\ .$$

$$\cdot\ {}_{g+H}F_{i+J}(a_g+p+\lambda,\, B_H+\lambda\,;\alpha_i+p+\lambda,\, C_J+\lambda\,;y)$$

$$= \sum_{\lambda=0}^{n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \binom{n}{\lambda} \frac{\Pi(a_g)_{\lambda+p+q}\, \Pi(b_h)_p\, \Pi(B_H)_{\lambda+q}\, x^p\, y^{q+\lambda}}{\Pi(\alpha_i)_{\lambda+p+q}\, \Pi(c_j)_p\, \Pi(C_J)_{\lambda+q}\, (B)_\lambda\, p!\, q!}$$

= Right hand side of (6).

Repeating the process we have (7).

To establish (8), we have

$$F\begin{matrix} g+1:h;H \\ \\ i+1:j;J\end{matrix}\left(\begin{matrix} a_g, a+n : b_h ; B_H ; \\ & x, y \\ \alpha_i, \quad a : c_j ; C_J ; \end{matrix}\right)$$

$$= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\Pi(a_g)_{p+q}\, (a+n)_{p+q}\, \Pi(b_h)_p\, \Pi(B_H)_q\, x^p\, y^q}{\Pi(\alpha_i)_{p+q}\, (a)_{p+q}\, \Pi(c_j)_p\, \Pi(C_J)_q\, p!\, q!}$$

$$= \sum_{p=0}^{\infty} \frac{\Pi(a_g)_p\, (a+n)_p\, \Pi(b_h)_p\, x^p}{\Pi(\alpha_i)_p\, (a)_p\, \Pi(c_j)_p\, p!}\ .$$

$$\cdot\ {}_{g+H+1}F_{i+J+1}(a_g+p,\, a+p+n,\, B_H\,;\alpha_i+p,\, a+p\,;C_J\,;y)$$

On using (3), this becomes

$$= \sum_{p=0}^{\infty} \frac{\Pi(a_g)_p\, (a+n)_p\, \Pi(b_h)_p\, x^p}{\Pi(\alpha_i)_p\, (a)_p\, \Pi(c_j)_p\, p!} \sum_{\lambda=0}^{n} \binom{n}{\lambda} \frac{\Pi\,(a_g+p)_\lambda\, \Pi(B_H)_\lambda\, y^\lambda}{\Pi(\alpha_i+p)_\lambda\, \Pi(C_J)_\lambda\, (a+p)_\lambda}\ .$$

$$\cdot\ {}_{g+H}F_{i+J}(a_g+p+\lambda,\, B_H+\lambda\,;\alpha_i+p+\lambda,\, C_J+\lambda\,;y)$$

$$= \sum_{q=0}^{\infty} \sum_{\lambda=0}^{n} \binom{n}{\lambda} \frac{\Pi(a_g)_\lambda\, \Pi(B_H)_{\lambda+q}\, y^\lambda}{\Pi(\alpha_i)_\lambda\, \Pi(C_J)_{\lambda+q}\, (a)_\lambda}\ .$$

$$\cdot\ {}_{g+h+1}F_{i+j+1}\,(a_g+q+\lambda,\, a+n\,,\, b_h\,;\alpha_i+\lambda+q\,,\, a+\lambda\,,\, c_j\,;x)$$

Using (3) again, we find it to be equal to the right hand side of (8).

Other results can similarly be obtained with the help of the various results obtained above.

Particular Cases

If in a ${}_pF_q$ form, k of the numerator parameters exceed k of the denominator parameters by positive integers say n_i $(i = 1, \ldots, k)$, the form ${}_pF_q$ may be expressed as the sum of $(n_1 + 1)(n_2 + 1)\ldots(n_k + 1)$ ${}_{p-k}F_{q-k}$.

The results[4] are particular cases of (3). Formula[5] can be deduced from (4) after some simplification.

As a particular case of (11), we have

$$F\begin{matrix} 1:h;H \\ \\ 1:j;J \end{matrix}\left[\begin{matrix} a+n:b_h;B_H; \\ \\ a:\ c_j;\ C_J; \end{matrix}\ x, y\right]$$

$$= \sum_{\lambda=0}^{n} \sum_{\mu=0}^{n-\lambda} \binom{n}{\lambda}\binom{n-\lambda}{\mu} \frac{\Pi(b_h)_\lambda\, \Pi(B_H)_\mu\, x^\lambda\, y^\mu}{\Pi(c_j)_\lambda\, \Pi(C_J)_\mu\, (a)_{\lambda+\mu}} \cdot$$

$$\cdot\ {}_hF_j(b_h + \lambda; c_j + \lambda; x)\quad {}_HF_J(B_H + \mu; C_J + \mu; y).$$

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REFERENCES

1. APPELL, P. & KAMPÉ, J. DE FÉRIET, "Fonctions Hypergeométrique, Hypersphérique", (Gauthier Villars, Paris), 1926.
2. RAGAB, F. M., *J. Reine Angew. Math.*, **212** (1963), 113.
3. RAINVILLE, E. D., "Special Functions", (Macmillan, New York), 1960.
4. LUKE, Y. L., "The Special Functions and their Approximations," Vol. 1, (Academic Press, New York), 1969, p. 110, (36) and (37).
5. *Ibid*, p. 111, (38).