VORTEXLINE GEOMETRY OF PSEUDOSTATIONARY GAS FLOWS

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The various kinetic and kinematic properties of inviscid unsteady gas flows, considering the geometric properties of a pseudostationary vortexline. Analytic expression for vorticity is obtained in terms of the components of the velocity. The compatibility conditions governing the flow are transformed in intrinsic form which form the feature of this investigation. Finally an attempt is made to study the complex-lamellar flows in geometric parameters of a pseudostationary vortexline.

BASIC EQUATIONS

The intrinsic properties[1-3] of pseudostationary gas flow have been studied. The basic equations governing pseudostationary gas flow, in the absence of viscosity and thermal conductivity in streaklines velocity vector field[3] are

$$\left.\begin{aligned} 3\rho + (\vec{Q}\cdot\nabla)\rho + \rho(\nabla\cdot\vec{Q}) &= 0 && (a)\\ \vec{Q} + (\vec{Q}\cdot\nabla)\vec{Q} + \rho^{-1}\nabla p &= 0 && (b)\\ \vec{Q}\cdot\nabla S &= 0 && (c)\\ \rho &= \rho(p, S) && (d) \end{aligned}\right\} \quad (1)$$

The compatibility conditions to be satisfied by the pseudostationary velocity vector are

$$\left.\begin{aligned} &\vec{Q}\,\vec{W} + \tfrac{1}{2}\nabla Q^2 \{\vec{W} + \operatorname{curl}\vec{W}\wedge\vec{Q}\} + \operatorname{curl}\,(\vec{W}\wedge\vec{Q})\,.\,(\vec{Q} + \vec{W}\wedge\vec{Q}) = 0\,. && (a)\\ &\operatorname{curl}\left[\frac{(3+\nabla\,.\,Q)\,\{\vec{Q} + (\vec{Q}\,.\,\nabla)\vec{Q}\}}{\vec{Q}\,\{\vec{Q} + (\vec{Q}\,.\,\nabla)\,\vec{Q}\}} + \vec{Q}\wedge\{\vec{W} + \operatorname{curl}\,\vec{W}\wedge\vec{Q}\}\right] = 0. && (b)\\ &\operatorname{curl}\{\vec{Q} + (\vec{Q}\,.\,\nabla)\,\vec{Q}\}\,.\,\nabla\left[\frac{\vec{Q}\{\vec{Q} + (\vec{Q}\,.\,\nabla)\vec{Q}\}}{3+\nabla\,.\,\vec{Q}}\right] = 0\,. && (c) \end{aligned}\right\} \quad (2)$$

GEOMETRIC RESULTS

Considering $\vec{t}, \vec{n}, \vec{b}$ as triply orthogonal unit tangent vectors, principal normals and binormals of the curves of congruences formed by pseudostationary vortexlines respectively and denoting $\frac{d}{ds}, \frac{d}{dn}, \frac{d}{db}$ as directional derivatives, along these vectors also selecting $\vec{r}$ as the position vector in space, we have the following geometric results[4]:

$$\frac{d\vec{r}}{ds} = \vec{t} = \frac{\vec{W}}{W} \quad (3)$$

$$\vec{Q} = \vec{t}\,Q_t + \vec{n}\,Q_n + \vec{b}\,Q_b \quad (4)$$

where Q_t, Q_n, Q_b are the resolved parts of the velocity components along pseudostationary vortexline principal normal and binormal respectively.

INTRINSIC DECOMPOSITION

I have decomposed the basic equations into intrinsic form and studied some of the interesting kinematic and kinetic properties of flows.

Operating curl on (4), we obtain

$$\left.\begin{aligned} & Q_t\,(\sigma'-\sigma'') + \frac{d\,Q_b}{dn} - \frac{d\,Q_n}{db} = W && (a)\\ & K''\,Q_b + \frac{d\,Q_t}{db} - Q_n\,(\tau+\sigma'') - \frac{d\,Q_b}{ds} = 0 && (b)\\ & K\,Q_t - K'\,Q_n + Q_b\,(\sigma'-\tau) + \frac{d\,Q_n}{ds} - \frac{d\,Q_t}{dn} = 0 && (c) \end{aligned}\right\} \tag{5}$$

These are the intrinsic relations to be satisfied by the pseudostationary vortexline geometry in velocity vector field components which are independent of the nature of the fluid compressible or incompressible.

Using (4) in (1 a), we have

$$\rho d + Q_t \frac{d\rho}{ds} + Q_n \frac{d\rho}{dn} + Q_b \frac{d\rho}{db} = 0 \tag{6}$$

where

$$d = 3 - (K' + K'')\,Q_t - K\,Q_n + \frac{d\,Q_t}{ds} + \frac{d\,Q_n}{dn} + \frac{d\,Q_b}{db} = 0$$

This expresses the conservation of mass along the congruences formed by the pseudostationary vortexlines, principal normals and binormals.

Also using the solenoidal property of $\vec{W}$ we obtain

$$K' + K'' = -J = \frac{d}{ds}\log W \tag{7}$$

From this we observe that the vorticity magnitude is uniform along an individual pseudostationary vortexline, if normal surfaces are minimal and the converse is also true.

Making use of (3) and (4) in (1b), we obtain

$$\left.\begin{aligned} & Q_t + Q\,\frac{dQ}{ds} + \rho^{-1}\,\frac{dp}{ds} = 0 && (a)\\ & Q_n - W\,Q_b + Q\,\frac{dQ}{dn} + \rho^{-1}\,\frac{dp}{dn} = 0 && (b)\\ & Q_b + W\,Q_n + Q\,\frac{dQ}{db} + \rho^{-1}\,\frac{dp}{db} = 0 && (c) \end{aligned}\right\} \tag{8}$$

These give us the conservation of momentum in pseudostationary vortexline geometry.

The energy equation (1 c) can be written as

$$Q_t \frac{dS}{ds} + Q_n \frac{dS}{dn} + Q_b \frac{dS}{db} = 0 \tag{9}$$

The compatibility conditions (2) can be decomposed in intrinsic form as

$$a\,(W + A) + b\,B + c\,C = 0 \tag{10}$$

$$\left.\begin{aligned} & (\sigma'-\sigma'')\,L + \frac{dN}{dn} - \frac{dM}{db} = 0 && (a)\\ & K''\,N - M\,(\tau+\sigma'') + \frac{dL}{db} - \frac{dN}{ds} = 0 && (b)\\ & KL - K'\,M + N\,(\sigma'-\tau) + \frac{dM}{ds} - \frac{dL}{dn} = 0 && (c) \end{aligned}\right\} \tag{11}$$

$$(W + A) \frac{d}{ds}\left(\frac{D}{d}\right) + B \frac{d}{dn}\left(\frac{D}{d}\right) + C \frac{d}{db}\left(\frac{D}{d}\right) = 0 \tag{12}$$

where $A = \frac{d}{dn}(W Q_n) + \frac{d}{db}(W Q_b)$

$$B = K'' W Q_n + W Q_b (\tau + \sigma'') - \frac{d}{ds}(W Q_n)$$

$$C = W Q_n (\sigma' - \tau) + K' W Q_b - \frac{d}{ds}(W Q_b)$$

$$a = Q_t + Q \frac{dQ}{ds}$$

$$b = Q_n - W Q_b + Q \frac{dQ}{dn}$$

$$c = Q_b + W Q_n + Q \frac{dQ}{db}$$

$$D = a Q_t + b Q_n + c Q_b$$

$$L = \frac{ad}{D} + C Q_n - B Q_b$$

$$M = \frac{bd}{D} + (W + A) Q_b - C Q_t$$

$$N = \frac{cd}{D} + B Q_t - (W + A) Q_n$$

COMPLEX-LAMELLAR FLOW

A complex-lamellar flow is one in which the streaklines are normal to a one parameter family of surfaces, i.e., it is a field of the type[5].

$$\vec{Q} = \alpha(r) \nabla \phi(r) \tag{13}$$

where $\phi(r)$ = constant are defined as Beltrami surfaces and the function $\alpha(r)$ is called the distance function for the family of the Beltrami surfaces.

Forming the scalar product of (3) and (4), we obtain $Q_t = 0$, i.e., the velocity vector lies in the normal plane of the pseudostationary vortexline for a complex-lamellar flow.

The vorticity expressions (7) simplify to

$$\left.\begin{aligned} &W = \frac{d Q_b}{dn} - \frac{d Q_n}{db} && (a) \\ &K'' Q_b - Q_n (\tau + \sigma'') - \frac{d Q_b}{ds} = 0 && (b) \\ &-K' Q_n + Q_b (\sigma' - \tau) + \frac{d Q_n}{ds} = 0 && (c) \end{aligned}\right\} \tag{14}$$

Operating curl on (13) we obtain

$$\vec{W} = \nabla \alpha \wedge \nabla \phi \tag{15}$$

This shows that the pseudostationary vortexline is the curve of intersection of the Beltrami and its distance function surfaces.

Equations (6), (8 a) and (9) simplify to

$$\rho d' + Q_n \frac{d\rho}{dn} + Q_b \frac{d\rho}{db} = 0 \tag{16}$$

where $d' = 3 - K\, Q_n + \frac{d\, Q_n}{dn} + \frac{d\, Q_b}{db}$

$$Q \frac{dQ}{ds} + \rho^{-1} \frac{dp}{ds} = 0 \tag{17}$$

$$Q_n \frac{dS}{dn} + Q_b \frac{dS}{db} = 0 \tag{18}$$

The compatibility conditions (10) to (12) simplify to

$$a' (W + A) + bB + cC = 0 \tag{19}$$

$$\left.\begin{aligned} &(\sigma' - \sigma'') L' + \frac{dN'}{dn} - \frac{dM'}{db} = 0 && (a)\\ &K'' N' - M' (\tau + \sigma'') + \frac{dL'}{db} - \frac{dN'}{ds} = 0 && (b)\\ &KL' - K' M' + N' (\sigma' - \tau) + \frac{dM'}{ds} - \frac{dL'}{dn} = 0 && (c) \end{aligned}\right\} \tag{20}$$

$$(W + A) \frac{d}{dS} \left(\frac{D'}{d'}\right) + B \frac{d}{dn} \left(\frac{D'}{d'}\right) + C \frac{d}{db} \left(\frac{D'}{d'}\right) = 0 \tag{21}$$

where $a' = Q \frac{dQ}{ds}$

$D' = b\, Q_n + c\, Q_b$

$L' = \frac{a' d'}{D'} + C\, Q_n - B\, Q_b$

$M' = \frac{bd'}{D'} + (W + A)\, Q_b$

$N' = \frac{cd'}{D'} - (W + A)\, Q_n$

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