SONIC DISCONTINUITIES IN A RADIATIVE GAS

S. N. OJHA & B. G. VERMA

Gorakhpur University, Gorakhpur

(*Received* 29 *March* 1971)

The behaviour of sonic discontinuities when a perfect gas is subjected to radiation has been studied. The growth and decay of plane and spherical waves has also been studied after obtaining second and third order compatibility.

When pressure, density and velocity are continuous across a moving surface $\Sigma (t)$, while at least one of the first derivatives of these quantities with respect to the space coordinates are discontinuous, Thomas[1] called it a sonic wave of order one or simply a sonic wave and discussed its growth and decay. In this paper, we have studied how these sonic discontinuities behave when a perfect gas is subjected to radiation. In the course of discussion we have introduced Klimshin's coefficient, obtained second and third order compatibility conditions and studied the growth and decay of plane and spherical waves.

EQUATIONS OF MOTION AND COMPATIBILITY CONDITIONS

Neglecting viscosity and thermal conductivity the differential equations governing the equation of motion of a perfect gas, when radiation effects are taken into accounts, are

$$\rho \frac{\partial u_i}{\partial t} + \rho u_i \, u_{i,j} + p_{,i} = 0 \tag{1}$$

$$\frac{\partial \rho}{\partial t} + u_i \, \rho_{,i} + \rho u_{i,i} = 0 \tag{2}$$

$$\frac{\partial p}{\partial t} - \rho u_i \frac{\partial u_i}{\partial t} - \rho u_i \, u_j \, u_{i,j} + kp \, u_{i,i} + (k-1) F_{i,i} = 0 \tag{3}$$

where

$$F_i = - \frac{ac}{\tau \rho} (p_R)_{,i} \text{ and } p_M = p + p_R \tag{4}$$

p_M and p_R being the material pressure and radiation pressure[2]. Equations (1), (2) and (3) are referred to a system of rectangular coordinates x_i, comma (,) indicates partial derivative with respect to these coordinates, F_i is the radiation flux and k is the generalized Klimshin coefficient[3] to be defined later. The material energy and radiation energy are given by $\frac{p_M}{\rho(\gamma-1)}$ and $\frac{3p_R}{\rho}$ respectively where γ is the ratio of specific heats. Assuming

$$p_M = zp \text{ and } p_R = (1-z)\, p \quad (0 < z < 1)$$

the total energy of the gas is given by

$$\frac{p}{\rho\,(k-1)}$$

where $$k = \{4\,(\gamma-1) + z\,(4-3\gamma)\} \,/\, \{3\,(\gamma-1) + z\,(4-3\gamma)\}$$

It is easy to see that when $z=1$, that is when radiation effects are not considered, the Klimshin coefficient k becomes equal to the usual adiabatic exponent γ. We can write,

$$F_i = \frac{A}{\rho} p_{,i} \tag{5}$$

where $$A = -\frac{ac(1-z)}{\tau}$$

c being the velocity of light, τ is the coefficient of opacity and a is a Stefan Boltzmann constant.

Let the moving surface be denoted by $\Sigma(t)$. Then if the discontinuity or jump across the moving surface is indicated by a bracket [ ], we have,

$$[p] = [\rho] = [u_i] = [F_i] = 0 \tag{6}$$

over $\Sigma(t)$. We assume the regularity of the surface $\Sigma(t)$ and the existence of the limiting values of the functions and their derivatives as one approaches this surface from each side. If G be the velocity of the moving surface, the following relations, called compatibility conditions of the first order, are satisfied.

$$[u_{i,j}] = \lambda_i \nu_j \qquad \left[\frac{\partial u_i}{\partial t}\right] = -G\lambda_i \tag{7}$$

$$[p_{,i}] = \xi\nu_i \qquad \left[\frac{\partial p}{\partial t}\right] = -G\xi \tag{8}$$

$$[\rho_{,i}] = \zeta\nu_i \qquad \left[\frac{\partial \rho}{\partial t}\right] = -G\zeta \tag{9}$$

$$[F_{i,j}] = \eta_i \nu_j \qquad \left[\frac{\partial F_i}{\partial t}\right] = -G\eta_i \tag{10}$$

where the quantities ξ, ζ, λ_i and η_i are suitable functions defined over the surface $\Sigma(t)$, ξ and ζ being scalars on $\Sigma(t)$. The quantities λ_i and η_i can be replaced by scalars λ and η since $\lambda_i = \lambda\nu_i$ and $\eta_i = \eta\nu_i$ where ν_i are the components of the unit normal ν to the surface $\Sigma(t)$.

VELOCITY OF THE MOVING SURFACE

From (1) to (3) and the compatibility conditions (7) to (10), we get,

$$\rho(u_n - G)\lambda_i + \xi\nu_i = 0 \tag{11}$$

$$(u_n - G)\zeta + \rho\lambda_i\nu_i = 0 \tag{12}$$

$$\rho(G - u_n)\lambda_i u_i - G\xi + kp\lambda_i\nu_i + (k-1)\eta = 0 \tag{13}$$

where u_n is the normal velocity, multiplying (11) in turn by u_i and ν_i we get

$$\rho(u_n - G)\lambda_i u_i + \xi u_n = 0 \tag{14}$$

and
$$\rho(u_n - G)\lambda_i\nu_i + \xi = 0 \tag{15}$$

Adding (13) and (14) we have,

$$\xi(u_n - G) + kp\lambda_i\nu_i + (k-1)\eta = 0 \tag{16}$$

Multiplying (15) by kp, (16) by $\rho(u_n - G)$ and subtracting we get,

$$\xi\rho(u_n - G)^2 - \xi kp + (k-1)\eta\rho(u_n - G) = 0 \tag{17}$$

As a consequence of (15), (17) can be written as

$$\xi\left\{\rho(u_n - G)^2 - kp - \frac{(k-1)\eta}{\lambda}\right\} = 0 \tag{18}$$

Let the speed S of the sonic wave defined by $S = (G - u_n)$ be different from zero. Then, if $\xi = 0$ on Σ (t it follows from (11) and (12) that $\zeta = 0$ meaning thereby that the surface is not a sonic wave of order one which is contrary to our assumptions. Hence $\xi \neq 0$ and we have from (18)

$$(u_n - G)^2 = \left\{\frac{kp}{\rho} = \frac{(k-1)\eta}{\rho\lambda}\right\} \tag{19}$$

Again from the equation (12), (15) and (19), we get

$$\zeta = \rho\lambda / (G - u_n) \tag{20}$$

$$\lambda = \xi / \rho(G - u_n) \tag{21}$$

and
$$\eta = \left\{ (u_n - G)^2 - \frac{kp}{\rho} \right\} \frac{\rho\lambda}{(k-1)} \tag{22}$$

If we assume that the sonic wave is propagated into a gas at rest within which the total pressure p and density ρ are constant ; $u_i = 0$ on the surface $\Sigma(t)$ and hence the speed of propagation of the wave is given by

$$G^2 = \left\{ \frac{kp}{\rho} + \frac{(k-1)\,\eta}{\rho\lambda} \right\} \tag{23}$$

and, then the equations (20) to (22) become

$$\zeta = \rho\,\lambda/G\,, \quad \xi = \rho\, G\, \lambda \text{ and } \eta\,(k-1) = 2B\rho\lambda \tag{24}$$

where
$$2B = \left\{ G^2 - \frac{kp}{\rho} \right\}$$

CONDITIONS OF COMPATIBILITY OF THE SECOND AND THIRD ORDERS

The conditions of the compatibility of second and third orders of the quantities p, ρ, u_i and F_i are applicable in equations (47) to (51). When $G =$ constant, the compatibility conditions of the second order for velocity component u_i are given by

$$[u_{i\cdot jk}] = \bar{\lambda}_i\, \nu_j\, \nu_k + g^{\alpha\beta}\, \lambda_{i,\alpha}\, (\nu_j\, x_{k,\beta} + \nu_k\, x_{j,\beta}) - \lambda_i\, g^{\alpha\beta}\, g^{\sigma\tau}\, b_{\alpha\sigma}\, x_{j,\beta}\, x_{k,\tau} \tag{25}$$

and

$$\left[\frac{\partial^2 u_i}{\partial x_j \partial t} \right] = \left(-G\,\bar{\lambda}_i + \frac{\delta\lambda_i}{\delta t} \right) \nu_j - G\, g^{\alpha\beta}\, \lambda_{i,\alpha}\, x_{j,\beta} \tag{26}$$

The corresponding conditions of compatibility for the functions p, ρ and F_i are given by

$$[p;_{ij}] = \bar{\xi}\, \nu_i \nu_j + g^{\alpha\beta}\, \bar{\xi}_{,\alpha}\, (\nu_i\, x_{j,\beta} + \nu_j\, x_{i,\beta}) - \xi\, g^{\alpha\beta}\, g^{\sigma\tau}\, b_{\alpha\sigma}\, x_{i,\beta}\, x_{j,\tau} \tag{27}$$

$$\left[\frac{\partial^2 p}{\partial x_i \partial t} \right] = \left(-G\,\bar{\xi} + \frac{\partial\xi}{\partial t} \right) \nu_i - G g^{\alpha\beta}\, \xi_{,\alpha}\, x_{i,\beta} \tag{28}$$

$$[\rho,_{ij}] = \bar{\zeta}\, \nu_i \nu_j + g^{\alpha\beta}\, \zeta_{,\alpha}\, (\nu_i\, x_{j,\beta} + \nu_j\, x_{i,\beta}) - \zeta\, g^{\alpha\beta}\, g^{\sigma\tau}\, b_{\alpha\sigma}\, x_{i,\beta}\, x_{j,\tau} \tag{29}$$

$$\left[\frac{\partial^2 \rho}{\partial x_i \partial t} \right] = \left(-G\,\bar{\zeta} + \frac{\partial\zeta}{\partial t} \right) \nu_i - G\, g^{\alpha\beta} \zeta_{,\alpha}\, x_{i,\beta} \tag{30}$$

$$[F_{i,jk}] = \bar{\eta}_i\, \nu_j\, \nu_k + g^{\alpha\beta}\, \eta_{i,\alpha}\, (\nu_j\, x_{k,\beta} + \nu_k\, x_{j,\beta})\ \eta_i\, g^{\alpha\beta}\, g^{\sigma\tau}\, b_{\alpha\sigma}\, x_{j,\beta}\, x_{k,\tau} \tag{31}$$

The third order compatibility conditions for the quantity p is given[4] by

$$[p,_{ijk}] = \bar{\xi}\, \nu_i\, \nu_j\, \nu_k + \xi_{,\alpha}\, g^{\alpha\beta}\, (\nu_j\, \nu_k\, x_{i,\beta} + \nu_i\, \nu_k\, x_{j,\beta} + \nu_i\, \nu_j\, x_{k,\beta}) - $$
$$- \xi\, b_{\alpha\sigma}\, g^{\alpha\beta}\, g^{\sigma\tau}\, (\nu_i\, x_{j,\beta}\, x_{k,\tau} + \nu_j\, x_{i,\beta}\, x_{k,\tau} + \nu_k\, x_{i,\beta}\, x_{j,\tau}) \tag{32}$$

where the quantities $\bar{\lambda}_i$, $\bar{\xi}$, $\bar{\zeta}$ and $\bar{\eta}_i$ are new functions defined on the surface $\Sigma(t)$ and $b_{\alpha\sigma}$ are the components of the second fundamental form of the surface. The relations (25), (27), (29), (31) and (32) are called the geometrical conditions of compatibility and (26), (28) and (30) are called the kinematical conditions of compatibility.

Since $x_{i,\beta}$ are the components of the vectors tangential to the surface $\Sigma(t)$, we have,

$$\lambda_i\, x_{i,\beta} = \lambda\, \nu_i\, x_{i,\beta} = 0 \tag{33}$$

and
$$\lambda_{i,\alpha}\, x_{i,\beta} = \lambda_{,\alpha}\, \nu_i\, x_{i,\beta} + \lambda\, \nu_{i,\alpha}\, x_{i,\beta}$$
$$= (\lambda\, \nu_i)_{,\alpha}\, x_{i,\beta} = -\lambda\, \nu_i\, x_{i,\alpha\beta}$$
$$= -\lambda\, \nu_i\, \nu_i\, b_{\alpha\beta} = -\lambda\, b_{\alpha\beta} \tag{34}$$

where $x_{i,\alpha\beta}$ are the components of the second co-variant derivative based on the metric of the surface $\Sigma(t)$. Contracting the indices i and j in (25) and using (33) and (34), we get

$$[u_{i,ik}] = (\lambda_i\, \nu_i)\, \nu_k + g^{\alpha\beta}\, \lambda_{i,\alpha}\, \nu_i\, x_{k,\beta} - 2\lambda\, \Omega\, \nu_k \tag{35}$$

where Ω is the mean curvature of the surface $\Sigma(t)$. But since

$$\lambda_{i,\alpha}\, \nu_i = (\lambda_i\, \nu_i)_{,\alpha} - \lambda_i\, \nu_{i,\alpha}$$
$$= \lambda_{,\alpha} - \lambda\, \nu_i\, \nu_{i,\alpha}$$
$$= \lambda_{,\alpha} \tag{36}$$

(35) becomes
$$[u_{i,ik}] = (\bar{\lambda}_i\, \nu_i - 2\,\lambda\,\Omega)\, \nu_k + g^{\alpha\beta}\, \lambda_{,\alpha}\, x_{k,\beta} \tag{37}$$

Now multiplying (37) by ν_k we get

$$[u_{i,ik}]\, \nu_k = \bar{\lambda}_i\, \nu_i - 2\,\lambda\,\Omega \tag{38}$$

similarly,
$$[F_{i,ik}]\, \nu_k = \eta_i\, \bar{\nu}_i - 2\eta\, \Omega \tag{39}$$

If P_1 and P_2 be the values of a quantity P on sides 1 and 2 respectively of the surface $\Sigma(t)$ the discontinuity in the product PQ is given by

$$[PQ] = Q_2\,[P] + P_2\,[Q] - [P]\,[Q]$$

If the gas is at rest on the side 2 of $\Sigma(t)$ and if the pressure and density are constant on this side of the surface as in section relating to the "velocity of the moving surface", then

$$[PQ] = -[P]\,[Q] \tag{40}$$

provided the quantities P and Q involve derivatives of the pressure p or density ρ as a factor or have as a factor, the velocity components u_i or their derivatives. Thus, we have

$$\left[\frac{\partial^2 u_i}{\partial x_j\, \partial t}\right] \nu_j = -G\bar{\lambda}_i + \frac{\partial \lambda_i}{\partial t} \tag{41}$$

$$\left[p_{,ij}\right] \nu_j = \bar{\xi}\, \nu_i + g^{\alpha\beta}\, \xi_{,\alpha}\, x_{i,\beta} \tag{42}$$

$$\left[\frac{\partial^2 p}{\partial x_i\, \partial t}\right] \nu_i = -G\bar{\xi} + \frac{\partial \xi}{\partial t} \tag{43}$$

$$\left[\frac{\partial^2 \rho}{\partial x_i\, \partial t}\right] \nu_i = -G\bar{\zeta} + \frac{\partial \zeta}{\partial t} \tag{44}$$

$$[\, p_{,ijk}\,]\, \nu_j\, \nu_k = \bar{\xi}\nu_i + g^{\alpha\beta}\, \xi_{,\alpha}\, x_{i,\beta} \tag{45}$$

And with the help of (40), we have

$$\left.\begin{aligned}
[\,\rho_{,i}\, u_{i,j}\,]\, \nu_j &= [\,\rho_{,j}\, u_{i,i}\,]\, \nu_j = \zeta\lambda \\
[\,p_{,j}\, u_{i,i}\,]\, \nu_j &= -\xi\lambda \\
[\,u_{i,j}\, u_{j,k}\,]\, \nu_i\, \nu_j &= -\lambda^2 \\
\left[\rho_{,j}\, \frac{\partial u_i}{\partial t}\right] \nu_i\, \nu_j &= G\zeta\lambda \\
\left[u_{i,j}\, \frac{\partial u_i}{\partial t}\right] \nu_j &= G\lambda^2
\end{aligned}\right\} \tag{46}$$

APPLICATION OF THE COMPATIBILITY CONDITIONS

Differentiating (1), (2), (3) and (5) with respect to x and observing that $u_i = 0$ on $\Sigma(t)$, we have

$$\left[\rho_{,j} \frac{\partial u_i}{\partial t}\right] + \rho \left[\frac{\partial^2 u_i}{\partial x_j \partial t}\right] + \left[p_{,ij}\right] + \rho \left[u_{i,j} u_{j,k}\right] = 0 \tag{47}$$

$$\left[\frac{\partial^2 \rho}{\partial x_j \partial t}\right] + \left[\rho_{,i} u_{i,j}\right] + \left[\rho_{,j} u_{i,i}\right] + \rho \left[u_{i,ij}\right] = 0 \tag{48}$$

$$\left[\frac{\partial^2 p}{\partial x_j \partial t}\right] - \rho \left[u_{i,j} \frac{\partial u_i}{\partial t}\right] + k \left[p_{,j} u_{i,i}\right] + kp \left[u_{i,ij}\right] + (k-1) \left[F_{i,ij}\right] = 0 \tag{49}$$

$$\rho [F_{i,j}] = A\, p_{,ij} \tag{50}$$

Again differentiating (50) we get,

$$[\rho_{,k} F_{i,j}] + \rho [F_{i,jk}] = A [p_{,ijk}] \tag{51}$$

By using the compatibility conditions of second and third order, in (47) to (51), we get

$$\left(\rho \frac{\partial \lambda}{\partial t} + \bar{\xi}\right) - \rho G \bar{\lambda}_i \nu_i = 0 \tag{52}$$

$$\frac{\partial \zeta}{\partial t} = 2\zeta\lambda + 2\rho\lambda\Omega + G\bar{\zeta} - \rho\bar{\lambda}_i \nu_i \tag{53}$$

$$\frac{\partial \xi}{\partial t} = (k+1)\,\xi\lambda + 2kp\lambda\Omega + 2(k-1)\,\eta\Omega + G\bar{\xi} - kp\bar{\lambda}_i \nu_i - (k-1)\,\bar{\eta}_i \nu_i \tag{54}$$

$$A\bar{\xi} = \rho\eta \tag{55}$$

$$\zeta\eta + A\bar{\xi} = \rho\bar{\eta}_i \nu_i \tag{56}$$

Substituting for kp in (54) from (23), we obtain

$$\frac{\partial \xi}{\partial t} = \frac{(k+1)}{2}\,\xi\lambda + \rho G^2 \lambda\Omega + \frac{(k-1)}{2\lambda}\left\{\eta\bar{\lambda}_i \nu_i - \lambda\bar{\eta}_i \nu_i\right\} \tag{57}$$

From (52) and (55), we have

$$\rho\left(\frac{\partial \lambda}{\partial t} + \frac{\eta}{A}\right) = \rho G \bar{\lambda}_i \nu_i \tag{58}$$

Again, from (55) and (56) we have

$$G\rho\bar{\eta}_i \nu_i = \rho\eta\,(\lambda + G) \tag{59}$$

With the help of (57), (58) and (59), we get

$$\frac{\partial \xi}{\partial t} = \frac{(k+1)}{2}\,\xi\lambda + \rho\, G^2 \lambda\Omega + \frac{(k-1)}{2\lambda}\left\{\frac{\eta}{G}\left(\frac{\partial \lambda}{t} + \frac{\eta}{A}\right) - \frac{\lambda\eta}{G}\left(\lambda + G\right)\right\} \tag{60}$$

Differentiating (24) with respect to time, we have

$$\frac{\partial \xi}{\partial t} = \rho G \frac{\partial \lambda}{\partial t}, \quad \frac{\partial \zeta}{\partial t} = \frac{\rho}{\partial t} \frac{\partial \lambda}{\partial t} \tag{61}$$

Simplifying (60) by making use of (61) and (24), we get

$$\frac{\partial \xi}{\partial t} = \left\{\frac{(k+1)\,a}{2\rho G} - \frac{a}{G^3 \rho}\right\}\xi^2 + Ga\xi\Omega + \frac{\xi b}{G} \tag{62}$$

where $$a = G^2 / (G^2 - B)$$

and $$b = \frac{2(a-1) - Aa}{A}$$

Again with the help of (60), we get from (62)

$$\frac{\partial \lambda}{\partial t} = \left\{ \frac{(k+1)\,a}{2} - \frac{a}{G^2} \right\} \lambda^2 + Ga\lambda\Omega + \frac{\lambda b}{G} \tag{63}$$

and

$$\frac{\partial \zeta}{\partial t} = \left\{ \frac{(k+1)\,Ga}{\rho} - \frac{a}{G\rho} \right\} \zeta^2 + aG\zeta\Omega + \frac{\zeta b}{G} \tag{64}$$

When sonic wave surfaces $\Sigma\ (t)$ are propagated into a quiescent gas, (62), (63) and (64) give the equation of the quantities ξ, λ and ζ along the normal trajectories of these surfaces. From these equations one can also predict the growth and decay of the sonic discontinuities associated with the wave surface $\Sigma\ (t)$.

Let $\Sigma\ (t_0)$ represent the sonic wave surface at the time t_0. Then if σ be the distance measured from $\Sigma\ (t_0)$ along the normal trajectories to the family of surfaces $\Sigma\ (t)$ in the direction of propagation, $\sigma = G\,(t - t_0)$ and the quantities λ, ξ and ζ are functions of the distance σ along each of the normal trajectories. Hence,

$$\frac{\partial \xi}{\partial t} = G\,\frac{d\xi}{d\sigma}, \quad \frac{\partial \lambda}{\partial t} = G\,\frac{d\lambda}{d\sigma}, \quad \frac{\partial \zeta}{\partial t} = G\,\frac{d\zeta}{d\sigma} \tag{65}$$

From (62), (63) and (64), we get,

$$\frac{d\xi}{d\sigma} = \left\{ \frac{k+1}{2G} - \frac{1}{G^3} \right\} \frac{a}{G\rho}\,\xi^2 + a\xi\Omega + \frac{\xi b}{G^2} \tag{66}$$

$$\frac{d\lambda}{d\sigma} = \left\{ \frac{k+1}{2G} - \frac{1}{G^3} \right\} a\lambda^2 + a\lambda\Omega + \frac{\lambda b}{G^2} \tag{67}$$

and

$$\frac{d\zeta}{d\sigma} = \left\{ \frac{k+1}{2G} - \frac{1}{G^3} \right\} \frac{aG^2}{\rho}\,\zeta^2 + a\zeta\Omega + \frac{\zeta b}{G^2} \tag{68}$$

As we shall see below, it is convenient to use (66), (67) and (68), in the discussion that follows.

PLANE AND SPHERICAL WAVES

In the case of plane waves $\Sigma\ (t)$, the mean curvature $\Omega = 0$ and then (66), (67) and (68), take the form,

$$\frac{d\xi}{d\sigma} = \left\{ \frac{k+1}{2G} - \frac{1}{G^3} \right\} \frac{a}{G\rho}\,\xi^2 + \frac{\xi b}{G^2} \tag{69}$$

$$\frac{d\lambda}{d\sigma} = \left\{ \frac{k+1}{2G} - \frac{1}{G^3} \right\} a\lambda^2 + \frac{\lambda b}{G^2} \tag{70}$$

and

$$\frac{d\zeta}{d\sigma} = \left\{ \frac{k+1}{2G} - \frac{1}{G^3} \right\} \frac{aG^2}{\rho}\,\zeta^2 + \frac{\zeta b}{G^2} \tag{71}$$

Integrating these equations, we have

$$\xi = \xi_0 \Big/ \left\{ 1 - \alpha \left(1 + \beta\,\frac{\xi_0}{\rho} \right) \right\} \tag{72}$$

$$\lambda = \lambda_0 / \ \{ 1 - \alpha\ (1 + \beta\,G\,\lambda_0) \} \tag{73}$$

and

$$\zeta = \zeta_0 \Big/ \left\{ 1 - \alpha \left(1 + \beta\,\frac{G^2\,\zeta_0}{\rho} \right) \right\} \tag{74}$$

where $\alpha = \left(1 - e^{-b\sigma/G^2} \right)$ and $\beta = \frac{aG}{b} \left(\frac{k+1}{2G} - \frac{1}{G^3} \right)$

ξ_0, λ_0 and ζ_0 being the values of the scalars ξ, λ and ζ at points of the surface $\Sigma(t_0)$ where $\sigma = 0$. From (5) we can see that the value of β is always negative. Now from (72), (73) and (74), we can discuss the following result. If ξ_0, λ_0 and ζ_0 are negative the quantities ξ, λ and ζ will approach zero as the distance $\sigma \to \infty$ whereas for positive values of ξ_0, λ_0 and ζ_0 these quantities become infinite for the value of σ given by,

$$\begin{aligned} \sigma &= \frac{G^2}{b} \log \left(1 + \frac{\rho}{\beta \xi_0} \right) \\ &= \frac{G^2}{b} \log \left(1 + \frac{1}{\beta G \lambda_0} \right) \\ &= \frac{G^2}{b} \log \left(1 + \frac{\rho}{\beta G^2 \zeta_0} \right) \end{aligned} \tag{75}$$

It follows from the equation (24) that if one of the quantities ξ_0, λ_0 or ζ_0 is negative or positive the other two will likewise be negative or positive. Also, the three ratios in (75) must have equal values. In the first case, when the scalars are negative the sonic discontinuities will decay or be damped out while in the second case when the quantities are positive, the sonic discontinuities will go until the wave finally terminates in a shock for the value of σ given in (75). If the sonic wave surfaces Σ (t) consist of a family of concentric spheres the mean curvature Ω is $-1/R$ where R denotes the radii of the spheres of the family provided R is assumed to increase with the time t. Replacing the distance σ by R in (66), (67) and (68), we have

$$\frac{d\xi}{dR} = \left(\frac{k+1}{2G} - \frac{1}{G^3} \right) \frac{a}{G\rho} \xi^2 + \xi \left(\frac{b}{G^2} - \frac{a}{R} \right) \tag{76}$$

$$\frac{d\lambda}{dR} = \left(\frac{k+1}{2G} - \frac{1}{G^3} \right) a\lambda^2 + \lambda \left(\frac{b}{G^2} - \frac{a}{R} \right) \tag{77}$$

and

$$\frac{d\zeta}{dR} = \left(\frac{k+1}{2G} - \frac{1}{G^3} \right) \frac{aG^2}{\rho} \zeta^2 + \zeta \left(\frac{b}{G^2} - \frac{a}{R} \right) \tag{78}$$

Integrating (75), (76) and (77), we get

$$\frac{1}{\xi} = \left(\frac{1}{G^3} - \frac{k+1}{2G} \right) \frac{ab^{a-1}\, 2\pi i}{\rho\, G^{2a-1}\, \Gamma a} R^a e^{-\frac{bR}{G^2}} \tag{79}$$

$$\frac{1}{\lambda} = \left(\frac{1}{G^3} - \frac{k+1}{2G} \right) \frac{ab^{a-1}\, 2\pi i}{G^{2(a-1)}\, \Gamma a} R^a e^{-\frac{bR}{G^2}} \tag{80}$$

and

$$\frac{1}{\zeta} = \left(\frac{1}{G^3} - \frac{k+1}{2G} \right) \frac{ab^{a-1}\, 2\pi i}{G^{2(a-2)}\, \Gamma a} R^a e^{-\frac{bR}{G^2}} \tag{81}$$

where the integration has been carried out with the help of Hankel's contour. As $R \to \infty$, ξ, λ and ζ tend to zero indicating that the sonic discontinuities are damped out whereas they become indefinitely large as $R \to 0$ showing that the sonic wave must degenerate into a spherical shock. This fact is borne by the Hankel's contour as well.

REFERENCES

1. Thomas, T. Y., *J. Math. Mech.*, 6 (4), (1957), 455.
2. Chandrasekhar, S., "An Introduction to the Study of Stellar Structure" (1969), p. 55.
3. Ojha, S.N., Communicated for publication (1970).
4. Edelen, D. G. B. & Thomas, T. Y., *Arch. Rat. Mech. Anal.*, 9 (3), 1962, p. 245.