# SOME NEW COORDINATE SYSTEMS IN A RIEMANNIAN SPACE

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This paper defines two new coordinate systems viz. pseudo-geodesic and pseudo-Riemannian. Spaces for which the equations of pseudo-geodesics admit a first integral have also been studied.

Consider a space $V_n$ of coordinates $x^i$ $(i=1,\ldots,n)$ and metric $g_{ij}\,dx^i\,dx^j$, immersed in a Riemannian $V_m$ of coordinates $y^\alpha$ $(\alpha=1,\ldots,m)$ and metric $a_{\alpha\beta}\,dy^\alpha\,dy^\beta$. Considering a congruence of curves $\lambda^\alpha_{\tau 1}$ given by

$$\lambda^\alpha_{\tau 1} = t_{\tau_1}{}^i\, y^\alpha_{;i} + \sum_\nu C_{\nu\tau 1}\, N^\alpha_{\nu 1}, \tag{1}$$

Pan[1] defined the relative first curvature vector of the curve $C$ of the subspace $V_n$ as follows:

$$\eta^i = p^i - \sum_{\nu,\tau} \overline{C}_{\nu\tau 1}\, K_{\nu 1}\;\; t_{\tau 1}{}^i + \sum_{\nu,\tau} \overline{C}_{\nu\tau 1}\;\; K_{\nu 1}\; t_{\tau 1 k}\;\; \frac{dx^k}{ds}\;\; \frac{dx^i}{ds} \tag{2}$$

He also defined pseudo-geodesic curves of the subspace as the curves for which relative first curvature vanishes at each and every point of the curve. Pan[1] obtained the differential equation of pseudo-geodesics in the following form :

$$\frac{d^2x^i}{ds^2} + U^i_{jk}\;\frac{dx^j}{ds}\;\;\frac{dx^k}{ds} = 0 \tag{3}$$

where

$$U^i_{jk} = \left\{ {i \atop jk} \right\} - \sum_{\nu,\tau} \overline{C}_{\nu\tau 1}\, \Omega_{\nu 1 jK} \left( t_{\tau 1}{}^i - t_{\tau 1 l}\;\frac{dx^l}{ds}\;\;\frac{dx^i}{ds} \right). \tag{4}$$

Using this relative connection $U^i_{jk}$, Upadhyay & Trivedi[2] defined the relative covariant derivative of a mixed tensor $X^i_j$ as follows:

$$X^i_{j:k} \underline{\underline{def}}\; \partial_k\, X^i_j + X^l_j\, U^i_{lk} - X^i_l\; U^l_{jk}.$$

## PSEUDO-GEODESIC COORDINATES

If $s$ is the arc length of a curve $C$ through a point $P_0$, measured from that point, then we have

$$x^i = \overset{i}{x_0} + \left(\frac{dx^i}{ds}\right)_0 s + \tfrac{1}{2}\left(\frac{d^2x^i}{ds^2}\right)_0 s^2 + \;\ldots\ldots \tag{5}$$

the subscript zero denoting that the function is to be evaluated at the point $P_0$. If $C$ is a pseudo-geodesic, the coefficient of $\frac{1}{2}s^2$ is equal to $-\underset{0}{U^i{}_{jk}}\,\xi^j\,\xi^k$, where

$$\xi^j = \left(\frac{dx^j}{ds}\right)_0. \tag{6}$$

Consequently in case of a pseudo-geodesic we have

$$x^i = x^i_0 + \xi^i\, s - \tfrac{1}{2}\, \underset{0}{U^i_{jk}}\, \xi^j\, \xi^k\, s^2 + \ldots\ldots \tag{7}$$

We shall now define a system of coordinates for which

$$\underset{0}{U^i_{jk}} = 0, \tag{8}$$

and we call such a system of coordinates as the pseudo-geodesic coordinate system with pole at $P_0$.

From the definition of relative covariant derivative it is clear that :

'At the pole of a pseudo-geodesic coordinate system the components of relative covariant derivative are ordinary derivatives'.

The condition that a system of coordinates be pseudo-geodesic, with pole at $P_0$, may be expressed in another form as follows :

If in the relation

$$\bar{U}^{..h}_{ij} \frac{\partial x^d}{\partial \bar{x}^h} = U^d_{ab} \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial x^b}{\partial \bar{x}^j} + \frac{\partial^2 x^d}{\partial \bar{x}^i \partial \bar{x}^j} \tag{9}$$

we interchange the $x$'s and the $\bar{x}$'s, we may write the relation (8) in the form

$$-\bar{U}^{d}_{ab} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} = \frac{\partial^2 \bar{x}^d}{\partial x^i \partial x^j} - U^h_{ij} \frac{\partial \bar{x}^d}{\partial x^h} = \left( \frac{\partial \bar{x}^d}{\partial x^i} \right)_{:j}. \tag{10}$$

If the $\bar{x}$'s are pseudo-geodesic coordinates with pole at $P_0$, the coefficients $\bar{U}^d_{ab}$ all vanish at this point and therefore also the function $(\partial \bar{x}^d/\partial x^i) : j$. Conversely if $(\partial \bar{x}^d/\partial x^i) : j$ all vanish at $P_0$, it follows from (9), (since the functional determinant $\partial \bar{x}/\partial x$ is not zero) that the relative connection $\bar{x}^d_{ab}$ all vanish at $P_0$, showing that the $x$'s are pseudo-geodesic coordinates. Thus :

*Theorem I*

'The necessary and sufficient condition that a system of coordinates be pseudo-geodesic with pole at $P_0$ is that $(\partial \bar{x}^d/\partial x^i) : j = 0$'.

Now we shall prove the existence of a pseudo geodesic coordinate system for any $V_n$ with an arbitrary pole at $P_0$.

Let $x^j$ be a general system of coordinates whose values at $P_0$ are $x^i_0$ and $\bar{x}^i$ another system of coordinates defined by

$$\bar{x}^i = a^i_k \, (x^k - x^k_0) + \tfrac{1}{2} a^i_h \, U^h_{jk} \, (x^j - x^j_0)(x^k - x^k_0), \tag{11}$$

where the coefficients $a^i_k$ are constants and the determinant $| a^i_k |$ is not zero. Then at the point $P_0$ we have

$$(\partial \bar{x}^i/\partial x^k)_0 = a^i_k \tag{12}$$

and

$$(\partial^2 \bar{x}^i/\partial x^j \partial x^k)_0 = a^i_h \; \underset{0}{U}{}^h_{jk} \tag{13}$$

Consequently, at $P_0$ the right hand side of (9) takes the form

$$a^i_h \; \underset{0}{U}{}^h_{jk} - a^i_h \; \underset{0}{U}{}^h_{jk} = 0,$$

and the conditions are therefore satisfied that the coordinates $\bar{x}^i$ be pseudo-geodesics with pole at $P_0$.

Now it is easy to prove that for an arbitrary curve $C$ in $V_n$ it is possible to choose coordinates which are pseudo-geodesics at every point of $C$.

Since we know that for a geodesic coordinate system with pole at $P_0$ we have

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_0 = 0, \tag{14}$$

therefore from (4) we have the following:

*Theorem II.*

'The necessary and sufficient condition for geodesic coordinates to become pseudo-geodesic coordinates is given by either of the following:

(i) the congruence be normal,

(ii) the curve be an asymptotic line'.

PSEUDO-RIEMANNIAN COORDINATES

Let $C$ be any pseudo geodesic through a given point $P_0$ and $s$ be its arc length measured from $P_0$ To each point $P$ of the pseudo-geodesic we assign coordinates $y^i$ such that

$$y^i = \xi^i s. \tag{15}$$

The quantities $\xi^i$ determine the particular pseudo-geodesic through $P_0$ ; and the value of $s$ then determines the point $P$ on this pseudo-geodesic. As there is a pseudo-geodesic from $P_0$ to any point of $V_n$, each point of the space has definite coordinates $y^i$ assigned to it. These are the pseudo-Riemannian coordinates referred to. We shall now show that these are particular type of pseudo-geodesic coordinates with pole at $P_0$.

If $\bar{U}^i_{jk}$ are the coefficients of relative connections calculated with respect to the $y$'s, the differential equations of the pseudo-geodesics of $V_n$ in terms of these coordinates are

$$\frac{d^2y^i}{ds^2} + \bar{U}^i_{jk} \frac{dy^j}{ds} \frac{dy^k}{ds} = 0. \tag{16}$$

By virtue of (14) and (15) we can easily obtain

$$\bar{U}^i_{jk} \xi^j \xi^k = 0 \tag{17}$$

and therefore

$$\bar{U}^i_{jk} y^j y^k = 0 \tag{18}$$

holds throughout the space.

Conversely if (17) are satisfied then (15) are satisfied by (14) and the $y$'s are pseudo-Riemannian coordinates. Thus we have the following:

*Theorem III*

'If $U^i_{jk}$ are the relative connection for a coordinate system $y$ a necessary and sufficient condition that these be pseudo-Riemannian coordinates is that the equations'

$$U^i_{jk} y^j y^k = 0 \tag{19}$$

hold throughout the space.

The equations (16) hold at $P_0$ for all pseudo-geodesics through that point, that is to say, for all directions $\xi^i$. Consequently the coefficients $\bar{U}^i_{jk}$ must vanish at that point, showing that the pseudo-Riemannian coordinates are pseudo-geodesic coordinates with pole at $P_0$.

By using the definitions of Riemannian and pseudo-Remannian coordinates we easily obtain the following:

*Theorem IV*

'The necessary and sufficient condition for the Riemannian coordinates to become pseudo-Remannian coordinates is given by either of the following:

(i) the congruence be normal,

(ii) the curve be an asymptotic line.

PSEUDO-GEODESICS OF A SPACE

If each integral of the equations (3) of the pseudo-geodesics of a space satisfies the condition

$$a_{i_1 \cdots i_r} \frac{dx^{i_1}}{ds} \cdots \frac{dx^{i_r}}{ds} = \text{Constant,} \tag{20}$$

the equations (3) are said to admit a first integral of $r$th order.

Now let us suppose that the tensor $a_{i_1 \ldots i_r}$ is symmetric in all the subscripts, then differentiating (19) relative covariantly with respect to $x^j$ and multiplying by $dx^j/ds$ and making use of

$$(dx^j/ds)\,(dx^i/ds) : j = 0 \tag{21}$$

we obtain

$$a_{i_1 \cdots i_r : j} \frac{dx^{i_1}}{ds} \cdots \frac{dx^{i_r}}{ds} \frac{dx^j}{ds} = 0 \tag{22}$$

Since the equations (21) must be satisfied identically, we must have

$$P\,(a_{i_1 \cdots i_r : j}) = 0 \tag{23}$$

where $P$ indicates the sum of the $(m+1)$ terms obtained by permuting the subscripts cyclically.

In particular, if (19) is of the first order, i.e., if

$$a_i\,(dx^i/ds) = \text{Constant,} \tag{24}$$

the condition (22) reduces to

$$a_{i:j} + a_{j:i} = 0, \tag{25}$$

i.e., the vector $a_i$ is a relative Killing vector[2]. Thus we have:

*Theorem V*

'If the equation of a pseudo-geodesic admits an integral of the first order then the covariant vector $a_i$ is a relative Killing vector.'

REFERENCES

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