GENERALIZED INTEGRAL TRANSFORM OF TWO VARIABLES

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The paper introduces generalization of Laplace transform of two variables. An inversion formula and certain theorems for the generalized transform, have been established.

Agarwal[1] and Sharma[2] have recently defined the G-function of two variables which has been represented by Bajpai[3] as

$$G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3;(q_1, q_2), q_3}\left[\begin{array}{c|l} x & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1}\int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(b_j - s) \prod_{j=1}^{n_1} \Gamma(1 - a_j + s) \prod_{j=1}^{m_2} \Gamma(d_j - t) \prod_{j=1}^{n_2} \Gamma(1 - c_j + t)}{\prod_{j=m_1+1}^{q_1} \Gamma(1 - b_j + s) \prod_{j=n_1+1}^{p_1} \Gamma(a_j - s) \prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + t) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - t)} \times$$

$$\times \frac{\prod_{j=1}^{n_3} \Gamma(1 - e_j + s + t)}{\prod_{j=n_3+1}^{p_3} \Gamma(e_j - s - t) \prod_{j=1}^{q_3} \Gamma(1 - f_j + s + t)} x^s y^t \, ds \, dt. \qquad (1)$$

The contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops if necessary, to ensure that the poles of

$$\Gamma(b_j - s), \qquad j = 1, 2, \ldots. m_1$$

lie to the right and the poles of

$$\Gamma(1 - a_j + s), \quad j = 1, 2, \ldots, n_1$$

and

$$\Gamma(1 - e_j + s + t), \quad j = 1, 2, \ldots\ldots, n_3$$

to the left of the contour. Similarly the contour L_2 is in the t-plane and runs from $-i\ \infty$ to $+i\infty$ with loops if necessary, to ensure that the poles of

$$\Gamma(d_j - t), \quad j = 1, 2, \ldots\ldots, m_2$$

lie to the right and the poles of

$$\Gamma(1 - c_j + t), \quad j = 1, 2, \ldots\ldots, n_2$$

and

$$\Gamma(1 - e_j + s + t), \quad j = 1, 2, \ldots\ldots n_3$$

lie to the left of the contour. Provided that

$$0 \leqslant m_1 \leqslant q_1,\ 0 \leqslant m_2 \leqslant q_2,\ 0 \leqslant n_1 \leqslant p_1,\ 0 \leqslant n_2 \leqslant p_2,\ 0 \leqslant n_3 \leqslant p_3$$

the integral converges if

$$(p_1 + q_1 + p_3 + q_3) < 2\,(m_1 + n_1 + n_3)\,,$$
$$(p_2 + q_2 + p_3 \;\; q_3) < 2\,(m_2 + n_2 + n_3)$$
$$|\arg x| < [m_1 + n_1 + n_3 - \tfrac{1}{2}\,(p_1 + q_1 + p_3 + q_3)]\,\pi,$$
$$|\arg y| < [m_2 + n_2 + n_3 - \tfrac{1}{2}\,(p_2 + q_2 + p_3 + q_3)]\,\pi$$

Here as well as in what follows (a_p) represents the sequence of parameters $a_1, a_2, \ldots, a_p$.

The classical Laplace transform of a function $f(x)$ is

$$\phi(p) = p\int_0^\infty e^{-px} f(x)\,dx\,, \quad R(p) > 0 \tag{2}$$

The Laplace transform as defined by Humbert[4], for function $f(x, y)$ of two variables is

$$\phi(p, q) = pq\int_0^\infty\int_0^\infty e^{-px}\, e^{-qy} f(x, y)\,dx\,dy, \quad R(p, q) > 0 \tag{3}$$

In the present paper we introduce a generalization of Laplace transform in two variables in the form

$$\phi\,(p, q) = pq\int_0^\infty\int_0^\infty G\,{}^{(m_1+1,\; m_2+1)\;;\;(0,\,0),\; n_3}_{(m_1,\, m_2),\; p_3\;;\;(m_1+1,\; m_2+1),\; q_3}\left[\begin{matrix}\lambda px\\ \\ \mu qy\end{matrix}\,\middle|\,\begin{matrix}a_1+\alpha_1, \ldots, a_{m_1}+\alpha_{m_1}\;;\; b_1+\beta_1, \ldots, b_{m_2}+\beta_{m_2}\\ (e_{p_3})\\ a_1, \ldots\, a_{m_1}, \rho_1\;;\; b_1, \ldots, b_{m_2}, \rho_2\\ (f_{q_3})\end{matrix}\right] f(x, y)\,dx\,dy. \tag{4}$$

where

$$R(p, q) > 0,\; p_3 + q_3 < 1 + 2n_3,$$
$$|\arg p| < [n_3 - \tfrac{1}{2}\,(p_3 + q_3 - 1)\,]\,\pi, \quad |\arg q| < [n_3 - \tfrac{1}{2}\,(p_3 + q_3 - 1)\,]\,\pi$$

and

$$(x)^{b_j}\;(y)^{d_i}\; f(x, y) \in L\,(0, \delta),\; \delta > 0, \quad j = 1, \ldots, m_1, \quad i = 1, \ldots m_2$$

We shall represent (4) symbolically as

$$\phi(p, q) \;\underset{\cdot\cdot}{\overset{G}{=\!=}}\; f(x, y)$$

Putting $n_3 = p_3 = q_3 = 0$ and using the relation

$$G\,{}^{(m_1,\, m_2)\;;\;(n_1,\, n_2),\; 0}_{(p_1,\, p_2),\; 0\;;\;(q_1,\, q_2),\; 0}\left[\begin{matrix}x\\ \\ y\end{matrix}\,\middle|\,\begin{matrix}(a_{p_1})\;;\; c_{p_2})\\ \hline (b_{q_1})\;;\; d_{q_2})\end{matrix}\right] = G\,{}^{m_1,\, n_1}_{p_1,\, q_1}\left[x\,\middle|\,\begin{matrix}(a_{p_1})\\ (b_{q_1})\end{matrix}\right] G\,{}^{m_2\,,\, n_2}_{p_2,\, q_2}\left[y\,\middle|\,\begin{matrix}(c_{p_2})\\ (d_{q_2})\end{matrix}\right] \tag{5}$$

(4) is reduced to

$$\phi\,(p, q) = pq\int_0^\infty\int_0^\infty G\,{}^{m_1+1,\, 0}_{m_1,\, m_1+1}\left[\lambda px\,\middle|\,\begin{matrix}a_1+\alpha_1, \ldots, a_{m_1}+\alpha_{m_1}\\ a_1, \ldots, a_{m_1}, \rho_1\end{matrix}\right] \times$$
$$\times\, G\,{}^{m_2+1,\, 0}_{m_2,\, m_2+1}\left[\mu qy\,\middle|\,\begin{matrix}b_1+\beta_1, \ldots, b_{m_2}+\beta_{m_2}\\ b_1, \ldots, b_{m_2}, \rho_2\end{matrix}\right] f\,(x, y)\,dx\,dy \tag{6}$$

When $\lambda = \mu = 1$, (6) yields Meijer-Laplace transform of two variables given by Jain[5].

Further when

$$\alpha_j = 0,\ j = 1, \ldots, m_1;$$
$$\beta_i = 0,\ i = 1, \ldots, m_2;$$
$$\rho_1 = \rho_2 = 0 \text{ and } \lambda = \mu = 1,$$

(6) is reduced to the form (3).

The generalization of Laplace transform of $f(x)$ as defined by Bhise[6] is

$$\phi(p) = p \int_0^\infty G^{m+1,0}_{m,\,m+1} \left[px \,\middle|\, \begin{matrix} \eta_1 + \alpha_1, \ldots, \eta_m + \alpha_m \\ \eta_1, \ldots, \eta_m, \rho \end{matrix} \right] f(x)\, dx. \tag{7}$$

Following integral is required in the proof of inversion formula.

$$\int_0^\infty \int_0^\infty e^{-\lambda x} x^{-\alpha} e^{-\mu y} y^{-\beta} G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3;(q_1, q_2), q_3} \left[\begin{matrix} \eta x \\ \\ \zeta y \end{matrix} \,\middle|\, \begin{matrix} (a_{p_1});(c_{p_2}) \\ (e_{p_3}) \\ (b_{q_1});(d_{q_2}) \\ (f_{q_3}) \end{matrix} \right] dx\, dy$$

$$= \lambda^{\alpha-1} \mu^{\beta-1} G^{(m_1, m_2);(n_1+1, n_2+1), n_3}_{(p_1+1, p_2+1), p_3;(q_1, q_2), q_3} \left[\begin{matrix} \frac{\eta}{\lambda} \\ \\ \frac{\zeta}{\mu} \end{matrix} \,\middle|\, \begin{matrix} \alpha, (a_{p_1}); \beta, (c_{p_2}) \\ (e_{p_3}) \\ (b_{q_1});(d_{q_2}) \\ (f_{q_3}) \end{matrix} \right] \tag{8}$$

$$\operatorname{Re} \lambda > 0,\ \operatorname{Re} \mu > 0,$$
$$(p_1 + q_1 + p_3 + q_3) < 2(m_1 + n_1 + n_3),$$
$$(p_2 + q_2 + p_3 + q_3) < 2(m_2 + n_2 + n_3)$$
$$|\arg \eta| < [m_1 + n_1 + n_3 - \tfrac{1}{2}(p_1 + q_1 + p_3 + q_3)]\pi,$$
$$|\arg \xi| < [m_2 + n_2 + n_3 - \tfrac{1}{2}(p_2 + q_2 + p_3 + q_3)]\pi.$$

Equation (8) is established by expressing the G-function of two variables on the left as (1), interchanging the order of integration which is justified due to the absolute convergence of integrals involved in the process and evaluating the inner integral with the help of (7) viz

$$\int_0^\infty e^{-pt} t^n\, dt = n!\ p^{-n-1},\quad \operatorname{Re} p > 0$$

INVERSION FORMULA

We now establish the following theorem which gives us a solution of the integral (4), solved for the unknown function $f(x,y)$ in terms of its image $\phi(p, q)$.

Theorem

If

$$\phi(p, q) = pq \int_0^\infty \int_0^\infty G^{(m_1+1, m_2+1);(0,0)\, n_3}_{(m_1, m_2),\ p_3;(m_1+1, m_2+1), q_3} \left[\begin{matrix} \lambda p x \\ \\ \mu q y \end{matrix} \right.$$

$$\left| \begin{matrix} a_1 + \alpha_1, \ldots, a_{m_1} + \alpha_{m_1}; b_1 + \beta_1, \ldots, b_{m_2} + \beta_{m_2} \\ (e_{p_3}) \\ a_1, \ldots, a_{m_1}, \rho_1; b_1, \ldots, b_{m_2}, \rho_2 \\ (f_{q_3}) \end{matrix} \right] f(x, y)\, dx\, ay,$$

then

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{x^{-h} y^{-k}}{G(\lambda; \mu)} F(h, k)\, dh\, dk \tag{9}$$

where

$$F(h, k) = \int_0^\infty \int_0^\infty e^{-px} p^{-h-1} e^{-qy} q^{-k-1} \phi(p, q)\, dp\, dq .$$

and

$$G(\lambda ; \mu) = G^{(m_1+1, m_2+1) ; (1, 1), n_3}_{(m_1+1, m_2+1), p_3 ; (m_1+1, m_2+1), q_3} \left[\begin{array}{c|l} \lambda & h, a_1+\alpha_1, \ldots, a_{m_1}+\alpha_{m_1} ; k, b_1+\beta_1, \ldots, b_{m_2}+\beta_{m_2} \\ & (e_{p_3}) \\ \mu & a_1, \ldots, a_{m_1}, \rho_1 ; b_1, \ldots, b_{m_2}, \rho_2 \\ & (f_{q_3}) \end{array} \right]$$

provided that

$$\mathrm{Re}(x) > 0, \quad \mathrm{Re}(y) > 0,$$

and the generalized Laplace transform of $f(x, y)$ exists.

Proof—

From (4) we have

$$F(h, k) = \int_0^\infty \int_0^\infty e^{-px} p^{-h-1} e^{-qy} q^{-k-1} \int_0^\infty \int_0^\infty pq\, G^{(m_1+1, m_2+1) ; (0, 0), n_3}_{(m_1, m_2), p_3 ; (m_1+1, m_2+1), q_3} \left[\begin{array}{c|l} \lambda px & a_1+\alpha_1, \ldots, a_{m_1}+\alpha_{m_1} ; b_1+\beta_1, \ldots, b_{m_2}+\beta_{m_2} \\ & (e_{p_3}) \\ \mu qy & a_1, \ldots, a_{m_1}, \rho_1 ; b_1, \ldots, b_{m_2}, \rho_2 \\ & (f_{q_3}) \end{array} \right] f(x, y)\, dx\, dy\, dp\, dq.$$

Changing the order of integration which is justified by virtue of de la Vallee Poussin's theorem[8] as the integrals involved in the process are absolutely convergent under the conditions stated earlier and evaluating the inner integral with the help of (8) we get

$$F(h, k) = G(\lambda ; \mu) \int_0^\infty \int_0^\infty x^{h-1} y^{k-1} f(x, y)\, dx\, dy .$$

Now applying Reed's[9] theorem II, the result (9) is proved.

FUNDAMENTAL THEOREMS

Now we give a few fundamental theorems for the generalized transform (4) defined above.

Theorem I.

If

$$\phi(p, q) \underset{\cdot\cdot}{\overset{G}{=}} f(x, y)$$

then

$$\phi\left(\frac{p}{a}, \frac{q}{b}\right) \underset{\cdot\cdot}{\overset{G}{=}} f(ax, by) \tag{10}$$

Theorem II.

If

$$\phi(p, q) \underset{\cdot\cdot}{\overset{G}{=}} f(x, y)$$

then
$$\int_p^\infty \int_q^\infty \phi(p,q)\ \frac{dp}{p}\ \frac{dq}{q} \ \underset{\cdot\,\cdot}{\overset{G}{=}} \int_0^x \int_0^y f(x,y)\ \frac{dx}{x}\ \frac{dy}{y} \tag{11}$$

Theorem III.

If
$$\phi(p,q) \ \underset{\cdot\,\cdot}{\overset{G}{=}}\ f(x,y)$$

then
$$\int_0^\infty \int_0^\infty \phi(p,q)\ \frac{dp}{p}\ \frac{dq}{q} \ \underset{\cdot\,\cdot}{\overset{G}{=}} \int_0^\infty \int_0^\infty f(x,y)\ \frac{dx}{x}\ \frac{dy}{y} \tag{12}$$

Theorem IV.

If
$$\phi(p,q) \ \underset{\cdot\,\cdot}{\overset{G}{=}}\ f(x,y)$$

then
$$\int_0^p \int_0^q \phi(p,q)\ \frac{dp}{p}\ \frac{dq}{q} \ \underset{\cdot\,\cdot}{\overset{G}{=}} \int_x^\infty \int_y^\infty f(x,y)\ \frac{dx}{x}\ \frac{dy}{y} \tag{13}$$

Theorem V.

If
$$\phi_i(p,q) \ \underset{\cdot\,\cdot}{\overset{G}{=}}\ f_i(x,y), \ i = 1, 2$$

then
$$\int_0^\infty \int_0^\infty \phi_1(p,q).\,f_2(p,q)\ \frac{dp}{p}\ \frac{dq}{q} = \int_0^\infty \int_0^\infty f_1(p,q).\,\phi_2(p,q)\ \frac{dp}{p}\ \frac{dq}{q} \tag{14}$$

provided that the integrals involved are absolutely convergent.

Proof—Theorem I is established on replacing first p by p/a, q by q/a and then x by ax and y by (by)

To prove (11), divide both the sides of (10) by ab and integrate with respect to a and b between the limits zero and unity.

Proceeding similarly as above for (11) and taking the limits of integration zero and infinity, the result (12) is proved. If the limits of integration are unity and infinity, the result (13) is proved.

Theorem V is established by putting the value of $\phi_1(p,q)$ on the left hand side and changing the order of integration.

Theorem VI.

If
$$f(x,y) = f_1(x)\, f_2(y), \ n_3 = p_3 = q_3 = 0$$

then
$$\phi(p,q) = \phi_1(p)\ \phi_1(q) \tag{15}$$

where

$$\phi_1(p) = p \int_0^\infty G^{m_1+1,\,0}_{m_1,\,m_1+1}\left(\lambda p x \,\middle|\, \begin{matrix} a_1+\alpha_1, \ldots, a_{m_1}+\alpha_{m_1} \\ a_1, \ldots, a_{m_1}, \rho_1 \end{matrix}\right) f_1(x)\, dx$$

$$\phi_1(q) = q \int_0^\infty G^{m_2+1,\,0}_{m_2,\,m_2+1}\left(\mu q y \,\middle|\, \begin{matrix} b_1+\beta_1, \ldots, b_{m_2}+\beta_{m_2} \\ b_1, \ldots, b_{m_2}, \rho_2 \end{matrix}\right) f_2(y)\, dy$$

EXAMPLES

Now we get the images of few functions in this transform using the theorems already established.

Example 1.

Taking $f(x, y) = x^{s-1} y^{t-1}$, using (15) and equation[7] (14), we get

$$\phi(p, q) = \frac{p^{-s+1} q^{-t+1} \prod_{j=1}^{m_1} \Gamma(a_j + s)\ \Gamma(\rho_1 + s) \prod_{j=1}^{m_2} \Gamma(b_j + t)\ \Gamma(\rho_2 + t)}{\lambda^s \mu^t \prod_{j=1}^{m_1} \Gamma(a_j + \alpha_j + s) \prod_{j=1}^{m_2} \Gamma(b_j + \beta_j + t)}$$

where

$$\begin{aligned} &\operatorname{Re}(a_j + s) > 0 \quad j = 1, \ldots, m_1 \\ &\operatorname{Re}(b_j + t) > 0 \quad j = 1, \ldots, m_2 \\ &\operatorname{Re}(\rho_1 + s) > 0, \operatorname{Re}(\rho_2 + t) > 0 \\ &(1 + 3m_1) > 0, (1 + 3m_2) > 0 \end{aligned}$$

$$\left|\arg p\right| < (1 + 3m_1)\frac{\pi}{2}, \quad \left|\arg q\right| < (1 + 3m_2)\frac{\pi}{2}$$

Hence

$$\frac{p^{-s+1} q^{-t+1} \prod_{j=1}^{m_1} \Gamma(a_j + s) \prod_{j=1}^{m_2} \Gamma(b_j + t)\ \Gamma(\rho_1 + s)\ \Gamma(\rho_2 + t)}{\lambda^s \mu^t \prod_{j=1}^{m_1} \Gamma(a_j + \alpha_j + s) \prod_{j=1}^{m_2} \Gamma(b_j + \beta_j + t)} \overset{G}{\underset{..}{=}} x^{s-1} y^{t-1} \tag{16}$$

Example 2.

Taking $f(x, y) = (1 - x)^{\delta - 1} x^{s-1} (1 - y)^{\gamma - 1} y^{t-1}$, using equation[7] (15) and (5), we get

$$\phi(p, q) = \Gamma \gamma \Gamma \delta\, p q\, G_{(m_1+1, m_2+1), 0; (m_1+2, m_2+2), 0}^{(m_1, m_2); (1, 1), 0} \left[\begin{matrix} p\lambda \\ q\mu \end{matrix}\right| \left.\begin{matrix} 1-s, a_1 + \alpha_1, \ldots, a_{m_1} + \alpha_{m_1};\ 1-t, b_1 + \beta_1, \ldots, b_{m_2} + \beta_{m_2} \\ \overline{a_1, \ldots}, a_{m_1}, \rho_1, 1 - s - \delta;\ b_1, \ldots, b_{m_2}, \rho_2, 1 - t - \gamma \end{matrix}\right]$$

where $\operatorname{Re}\delta > 0, \operatorname{Re}\gamma > 0, (1 + 3m_1) > 0, (1 + 3m_2) > 0,$

$|\arg p| < (1 + 3m_1)\pi/2$, $|\arg q| < (1 + 3m_2)\pi/2$

Hence

$$\Gamma \delta \Gamma \gamma\, p q\, G_{(m_1+1, m_2+1), 0; (m_1+2, m_2+2), 0}^{(m_1, m_2); (1, 1), 0} \left[\begin{matrix} p\lambda \\ q\mu \end{matrix}\right| \left.\begin{matrix} 1-s, a_1 \times \alpha_1, \ldots, a_{m_1} + \alpha_{m_1};\ 1-t, b_1 + \beta_1, \ldots, b_{m_2} + \beta_{m_2} \\ \overline{a_1, \ldots}, a_{m_1}, \rho_1, 1 - s - \delta;\ b_1, \ldots, b_{m_2}, \rho_2, 1 - t - \gamma \end{matrix}\right]$$

$$\overset{G}{\underset{..}{=}} (1 - x)^{\delta - 1} (1 - y)^{\gamma - 1} x^{s-1} y^{t-1} \tag{17}$$

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