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Construction of Certain Infinite Families of Graceful Graphs from a Given Graceful Graph

B. D. ACHARYA

The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad-211 002

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Abstract. Two general methods of constructing an infinite sequence ($G = G_1, G_2, \ldots\ldots$) of graceful graphs G_i such that G_i is contained in G_{i+1} as an induced subgraph, for every given gracefully numbered graph G, is described along with applications of the concept of graceful graphs.

1. Introduction

The idea of graceful graphs was first introduced by Rosa[1] in 1966, and the name 'graceful graphs' was given by S. W. Golomb[2] who rediscovered them in 1968 as a generalization of the problem of giving a graceful numbering to an arbitrary tree whose possibility was at that time an unpublished but widely circulated conjecture, attributed to Gerhard Ringel[2], but now widely known in graph theory as the Ringel-Kotzig conjecture[3,4], which is stated as : 'All trees are graceful'. Golomb[2] also gave the following practical context for the problem : 'Think of the graph G as a communication network with n terminals and e interconnections between terminals. We wish to assign a distinct identifying number to each terminal, in such a way that each interconnection is then uniquely identified by the absolute value of the difference between the numbers assined to its two end terminals. For economy, the largest number assigned to any node is to be minimized'. This is clearly the same problem as that of finding a numbering of G. In communication network engineering this problem has now come to be known as the communication network labeling problem, and a numbering of G is called an efficient addressing and identification system[5] for G. Of late, a wide variety of practical contexts, such as X-ray crystallography, radioastronomy and missile guidance, where indexed graphs are used as models, have been cited[5,6,7,8,9].

For standard terminology and notation in graph theory, we follow F. Harary[10]. An indexer of a graph $G = (V, E)$ is an injection $f: V(G) \rightarrow N$ from the point set $V = V(G)$ of G into the set N of non-negative intergers. Let I_G denote the set of indexers of G. We call $f \epsilon I_G$ strong if the induced function $g_f : E(G) \rightarrow N$ on the line set $E = E(G)$ of G defined by

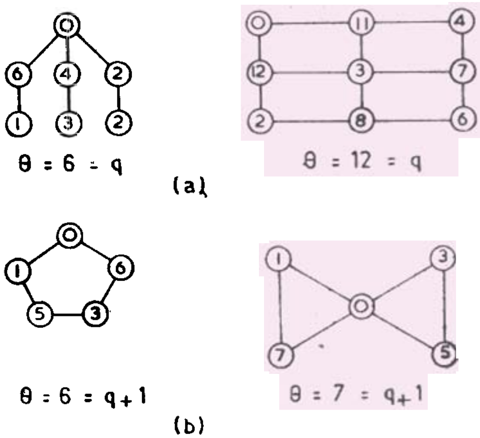

Figure 1(a). Two gracefully numbered graphs.

Figure 1(b). Two numbered non-graceful graphs.

$$g_f(u\,v) = |\,f(u) - f(v)\,| \quad uv \epsilon E(G) \tag{1}$$

is also injective, where $|\,r\,|$ denotes the absolute value of the real number r (when r is a set, we use the same notation for the cardinality of r). The set of strong indexers of G will be denoted I^*_G. Next for any $f \epsilon I_G$ let,

$$\left.\begin{aligned} f(G) &= \{f(u)/u\epsilon V(G)\} \\ g_f(G) &= \{g_f(e)/e\epsilon E(G)\} \end{aligned}\right\} \tag{2}$$

$$M(f) = \max_{u\epsilon V(G)} f(u). \tag{3}$$

Then the number

$$\theta(G) = \min_{f\epsilon I^*_G} M(f) \tag{4}$$

It is called the index of gracefulness of G. Evidently

$$\theta(G) \geqslant q(G), \tag{5}$$

where $q(G) = |\,E(G)\,|$. The graph G is said to be graceful if $\theta(G) = q(G)$, otherwise it is said to be non-graceful[2]. We let $N_G = \{f\epsilon I^*_G / M(f) = \theta(G)\}$. The elements of N_G are called numberings of G. The numberings of a graceful graph are specially called graceful numberings of G. For convenience, by the ordered pair (G, f), where G is a graph and $f\epsilon N_G$, we mean a numbered graph (or gracefully numbered graph if G is graceful). Fig. (1) displays two graceful and two non-graceful graphs each of which is given a numbering.

However, no characterization of graceful graphs has been discovered so far. The following two theorems are the only general results known on graceful graphs.

Theorem 1.1. A necessary condition[2] for a (p, q)-graph G to be graceful is that it be possible to partition the point set of G into two sets ζ and Θ such that the number of lines connecting points in ζ with points in Θ is exactly $\ulcorner(q+1)/2\urcorner$, where $\ulcorner r \urcorner$ denotes the integral part of the real number r.

Theorem 1.2. Every connected graph can be embedded as an induced subgraph in a connected graceful graphs.

While Theorem 1.1 gives a necessary condition for a graph to be graceful, Theorem 1.2 claims the impossibility of having a characterization of graceful graphs by means of forbidding a class of graphs to be induced subgraphs.[1]

To date, several infinite families of graceful, as well as non-graceful, graphs have been discovered[2,4,6,7,11–15]. A dictionary of graceful graphs and their graceful numberings is bound to be very much useful for the practitioner in building models of a real situation amenable for structural modeling.

The purpose of this paper is to describe two methods of constructing an infinite family of connected graceful graphs imbedding the given gracefully numbered graceful graph as an induced subgraph. Such constructions are useful in expanding the existing facility of graceful addressing and identification system on a communication network to a larger network obtained by introducing new user terminals and new communication links between the new terminals and the terminals of the original network.

2. Full Augmentation of a Graceful Graph

Every graceful (p, q) graph G can be embedded in a graceful $(q+1, q)$ graph[3] H. This may be achieved as follows: Let $f \epsilon N_G$. Then $\{1,\ldots\ldots q\} - f(G)$ has $m(G) = q - p + 1$ elements each of which does not appear as a node number in (G, f) (notice that $m(G)$ is nothing but the cycle rank[10] of G. Then adjoin $m(G)$ isolated points to G and assign them the integers from $\{1\ldots, q\} - f(G)$. For example, K_4 can be gracefully numbered as shown in Fig. 2. (a), has $m(K_4) = 3$ and can be augmented by $\bar{K}_3$ as shown in Fig. 2(b) to give a disconnected graceful graph. Clearly, addition of $m(G)$ isolates does not change the number of lines or their labeling from the original labeling.

In this context, $m(G)$ is called the node-deficit of G, and this deficit is reduced to zero in the new graph $G_f = G \cup \bar{K}_m$, where $m = m(G)$. Every graph with q lines and $q+1$ points is considered fully augmented[3].

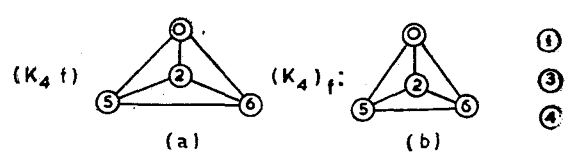

Figure 2. Augmentation of (K_4, f) in (a) is $(K_4)_f$ in (b).

3. Method of Joining a Totally Disconnected Graph

A (p, O) graph is called a totally disconnected graph and it may be written as $\bar{K}_p$. Given a gracefully numbered graph (G, f), we shall show in this section that the join[10] $\bar{K}_n + G_f$ is graceful for all $n \geqslant 1$, where G_f is the full augmentation of G as defined in Section 2.

Theorem 3. 1 Let (G, f) be a gracefully numbered (p, q) graph and let G_f be its full augmentation. Then for any integer $n \geqslant 1$, the graph $\bar{K}_n + G_f$ is graceful.

Proof: Let the points of $\bar{K}_n$ be labeled $x_1, \ldots, x_n$ and let the points of G_f be labeled $a_1, \ldots, a_p, b_1, \ldots, b_m$ where $a_1, \ldots, a_p$ are the p points of G and $b_1, \ldots, b_m$ are the $m = m(G)$ isolates adjoined to G. Let the numbers assigned to the points $b_1, \ldots, b_m$ be $k_1, \ldots, k_m$ respectively. Then define an assignment F of non-negative integers to the points of $\bar{K}_m + G_f$ as follows :

$$\left.\begin{aligned} F(a_i) &= f(a_i), && 1 \leqslant i \leqslant p \\ F(b_j) &= k_j && 1 \leqslant j \leqslant m \\ F(x_r) &= q + r(q + 1), && 1 \leqslant r \leqslant n. \end{aligned}\right\} \quad (6)$$

It is not hard to see that F is a graceful numbering of $\bar{K}_n + G_f$. ||

The numbering F defined in Eqn. 6 is displayed on $\bar{K}_3 + (K_4)_f$ in Fig. 3 where $(K_4)_f$ is the full augmentation of K_4 shown in Fig. 2(b).

Corollary 3.1.1: For any graceful tree T, the join $\bar{K}_n + T$ is graceful for all $n \geqslant$

Proof: This follows from Theorem 3.1 because $m(T) = 0$. ||

Remark : If the Rengel-Kotzig conjecture is true, Corollary 3.1.1 implies that for any tree T, the graph $\bar{K}_n + T$ is graceful for all $n \geqslant 1$.

Thus, for example, we have results like

*P*1: For any caterpillar[1,15] C, $\bar{K}_n + C$ is graceful for all $n \geqslant 1$.

*P*2: For any n-ary tree $T(n)$, the graph $\bar{K}_d + T(n)$ is graceful for all $d \geqslant$

Caterpillars and n-ary trees are known to be graceful[1,14].

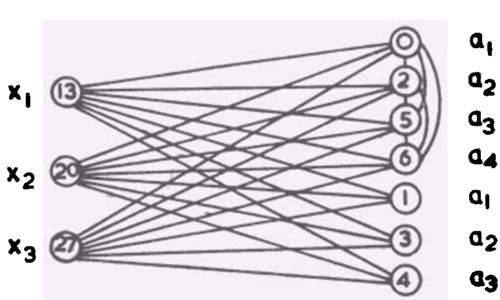

Figure 3. A graceful numbering of $K_3 + (K_4)_f$.

4. Method of Recursively Joining the Trivial Graph

Let (G, f) be a gracefully numbered (p, q) graph and G_{f_1} denote its full augmentation. Then the graph $K_1 + G_{f_1}$ becomes a gracefully numbered graph, if we assign the integer $q + (q + 1) = 2q + 1$ to the point of K_1. Let the full augmentation of this new gracefully numbered graph be denoted G_{f_2}. Again, the graph $K_1 + G_{f_2}$ becomes a gracefully numbered graph, if we assign the integer $2q + 1 + (2q + 1 + 1) = 4q + 3$ to the new point adjoined to G_{f_2} (i.e. the point of K_1). We may continue this procedure indefinitely. For any integer $i \geqslant 2$, define G_{f_i} to be the full augmentation of the gracefully numbered graph $K_1 + G_{f_{i-1}}$ in which the point of K_1 is assigned the number $2^{i-1} \{q(G) + 1\} - 1$ which is the number of lines in $K_1 + G_{f_{i-1}}$.

The above construction may be neatly expressed in terms of what is known as the Euclidean model of a numbered graph[2]. Euclidean model of a numbered graph (G, f) is obtained by placing the numbered points of G on the corresponding positions along the real axis and connecting them as in G. For instance, the Euclidean model of the second numbered graph shown in Fig. 1(b) is displayed in Fig. 4.

The following theorem gives the Euclidean model of $K_1 + G_{f_{i-1}}$ described above.

Theorem 4.1. Let (G, f) be a gracefully numbered (p, q) graph and G_{f_1} be its full augmentation. Then for each $i \geqslant 2$, the gracefully numbered graph $K_1 + G_{f_{i-1}}$ described above has for its nodes the first $2^{i-2}\{q(G) + 1\} - 1$ integral points on the

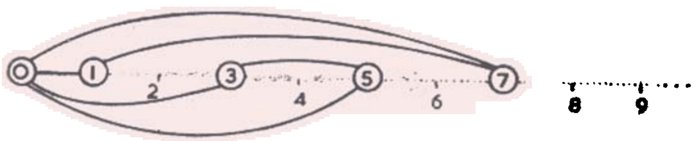

Figure 4. Euclidean model of the second numbered graph shown in Fig. 1(b).

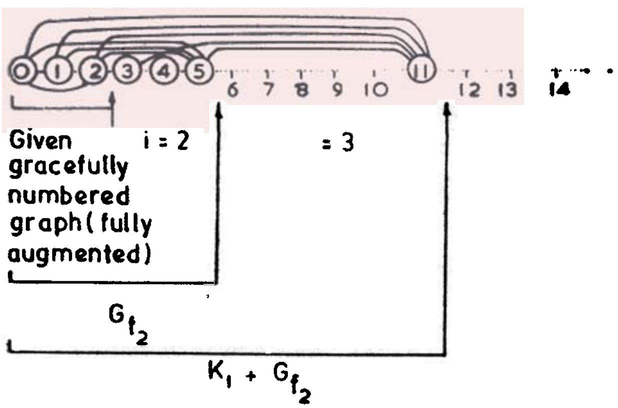

Figure 5. Euclidean model of $(K_1 + G_{f_2}.)$

real axis as also the point $2^{i-1}\{q(G) + 1\} - 1$, and has for its lines all the lines of G as also the lines joining the point $2^{i-1}\{q(G) + 1\} - 1$ to each integral point $j \leqslant 2^{i-2}\{q(G) + 1\} - 1$ on the real axis.

This theorem is illustracted in Fig. 5 taking a gracefully numbered P_3 (the path of length 2) and setting $i = 3$.

5. Conclusion

Given a gracefully numbered connected graceful graph G, we have given two methods of constructing an infinite sequence $(G = G_1, G_2, \ldots)$ of gracefully numbered connected graceful graphs G_i such that G_i is contained in G_{i+1} as an induced subgraph, $i = 1, 2, \ldots$. In view of Theorem 1.2, even if G is non-graceful, we can obtain an infinite sequence $(H = G_1, G_2, \ldots)$ of gracefully numbered graphs G_i with the said property by any one of the two methods described, where H contains G as an induced subgraph.

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