SOME INTEGRALS INVOLVING BESSELS FUNCTIONS AND FOX'S *H*-FUNCTION

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Some integrals involving Bessels functions and Fox's *H*-function have been evaluated. On specialising the parameters, the integrals yield many results for *G*-function and other related functions.

In this paper we have evaluated certain integrals involving Bessels functions and Fox's *H*-function by expressing the *H*-function as Mellin-Barnes type integral and interchanging the order of integrations. Some particular cases have been deduced.

Fox[1] introduced the *H*-function in the form of Mellin-Barnes type integral as

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p, c_p)\\(b_q, f_q)\end{matrix}\right]=\frac{1}{2\Pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)\,.\,z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)}\,.\,ds\,, \quad (1)$$

where z is not equal to zero and empty product is interpreted as unity; $p, q\ m$ and n are integers satisfying

$$1 \leqslant m \leqslant q,\ 0 \leqslant n \leqslant p;$$

$$a_j\,(j=1,\,\ldots\ldots\ldots,\,p)\,;\,f_j\,(j=1,\,\ldots\ldots\ldots,\,q)$$

are positive numbers and

$$a_j\,(j=1,\,\ldots\ldots\ldots,\,p)\,;\ b_j\ (j=1,\,\ldots\ldots\ldots,\,q)$$

are complex numbers such that no pole of

$$\Gamma\,(b_h-f_h)\,(h=1,\,\ldots\ldots\ldots,\,m)$$

coincides with any pole of

$$\Gamma\,(1-a_i+e_i s)\ (i=1,\,\ldots\ldots\ldots,\,n)\,,$$

$$e_i\,(b_h+\nu)\neq(a_i-\eta-1)\,f_h\quad(\nu\,,\eta=0\,,1\,,\,\ldots\ldots\ldots;\,h=1\,,\,\ldots\ldots,\,m,\,L=1\,,\,\ldots\ldots,\,n)\quad (2)$$

Further L runs from $\sigma-i\ \infty$ to $\sigma+i\ \infty$ such that the points

$$S=\frac{b_h+\nu}{f_h}\,(h=1\,,\,\ldots\ldots\ldots,\,m\,;\,\nu=0\,,1\,,\,\ldots\ldots\ldots)\,, \quad (3)$$

which are poles of

$$\Gamma\,(b_h-f_h)\,(h=1\,,\,\ldots\ldots\ldots,\,m)$$

on the right and the points :

$$S=\frac{a_i-\eta-1}{e_i}\quad(i=1\,,\,\ldots\ldots\ldots,\,n\,;\,\eta=0\,,1\,,\,\ldots\ldots\ldots) \quad (4)$$

which are the poles of

$$\Gamma\,(1-a_i+e_i s)\quad(i=1\,,\,\ldots\ldots\ldots,\,n)$$

lie on the left of L.

Recently Braaksma[2] has discussed the asymptotic expansion and analytic continuation of the *H*-function.

In what follows for the sake of brevity

$$\sum_{j=1}^{p} e_j-\sum_{j=1}^{q} f_j\equiv A\,,\ \sum_{j=1}^{n} e_j-\sum_{j=n+1}^{p} e_j+\sum_{j=1}^{m} f_j-\sum_{j=m+1}^{q} f_j\equiv B\,,$$

$(a_p\,,\,e_p$ denotes $(a_1\,,e_1)\,,\,\ldots\ldots\ldots,\,a_p\,,\,e_p)$.

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INTEGRALS

The first integral to be proved is

$$\int_0^a x^{2\rho-1} P_\mu (2x^2 \alpha^{-2} - 1) J_\nu (x)\ H_{p,q}^{m,n} \left[z.\ x^{2\delta} \left| \begin{matrix} (a_p\ ,\ e_p) \\ (b_q\ , f_q) \end{matrix} \right. \right] . dx$$

$$= \frac{2^{-\nu-1}\ \alpha^{2\rho+\nu}}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - \mu)} . \sum_{r=0}^{\infty} \frac{(-\alpha\, 2/4)^r}{\Gamma(\nu+1+r) r!} \times$$

$$\times H_{p+3,\, q+2}^{m,\, n+3} \left[z.\ \alpha^{2\delta} \left| \begin{matrix} (1+\mu-\nu/2-\rho, \delta), (1-r-\nu/2-\rho, \delta), (1-r-\nu/2-\rho, \delta), (a_p, e_p)\ ; \\ (b_q, f_q), (-\mu-r-\nu/2-\rho, \delta), (1+\mu-r-\nu/2-\rho, \delta, \end{matrix} \right. \right] \quad (5)$$

where δ is a positive number and

$$A \leqslant 0,\ B > 0,\ |\arg z| < B.\ \pi/2\ ,$$
$$\mathrm{Re}\,(\rho + \nu + \delta\, b_j / f_j) < 0\ ,\ \mathrm{Re}\,(2\rho + 2\delta\, b_j / f_j) > 0\ ,\ j = 1\ ,\ \ldots\ldots\ldots\ ,\ m;$$
$$\mathrm{Re}\,(\, 2\rho + 2\delta\, a_j / e_j\,) < 2\delta \quad (j = 1\ ,\ \ldots\ldots\ldots\ ,\ n\,).$$

Proof—

To prove (5) substituting from (1) on the left hand side of (5) we get

$$\int_0^a x^{2\rho-1} P_\mu (2x^2 \alpha^{-2} - 1) J_\nu (x) \left[\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + e_j s) .\ z^s .\ x^{2\delta s}}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^{p} \Gamma(a_j - e_j s)} \right] . dx$$

Interchanging the order of integration, which is justified due to the absolute convergence of the integrals involved in the process and on applying[3] [equation (32)], the expression reduces to the form

$$\frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + e_j s) .\ z^s .\ 2^{-\nu-1} .\ a^{2\rho + 2\delta s + \nu}}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^{p} \Gamma(a_j - e_j s)\ \Gamma(\frac{1}{2} + \frac{1}{2}\nu - \mu)} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\left\{ \Gamma\left(\frac{2\rho+\nu}{2} + r + \delta s \right) \right\}^2 \Gamma\left(\frac{2\rho+\nu}{2} - \mu + \delta s \right) \left(-\alpha\ \frac{2}{4} \right)^r}{\Gamma(\nu+1+r)\ \Gamma\left(\frac{2\rho+\nu}{2} + \mu + 1 + r + \delta s \right) \Gamma(\,\rho + \nu/2 + r - \mu + \delta s\,)\, r\,!} . ds$$

Now applying (1), the integral has the value given in the right hand side of (5).

On applying the same procedure and using[3] [equation (7)], the following integral is obtained;

$$\int_0^\infty x^{2\rho-1} J_\lambda (a\,x)\, J_\mu (a\,x)\, J_\nu\ (2b\,x)\, H_{p,q}^{m,n} \left[z.\ \ x^{2\delta} \left| \begin{matrix} (a_p,\ e_p) \\ (b_q,\ f_q) \end{matrix} \right. \right] dx$$

$$= \frac{a^{\lambda+\mu} . b^{-\lambda-\mu-2\rho}}{2^{\lambda+\mu}} . \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda+\mu+1}{2} \right)_r \left(\frac{\lambda+\mu}{2} + 1 \right)_r \left(\frac{a^2}{b^2} \right)^r}{\Gamma(\lambda+1+r)\ \Gamma(\mu+1+r)\ (\lambda+\mu+1)_r\, r\,!} \times$$

$$\times H_{p+3,\, q+1}^{m,\, n+2} \left[\frac{z}{b^{2\delta}} \left| \begin{matrix} \left(1-\rho-r-\frac{\lambda+\mu+\nu}{2}, \delta \right), \left(1-\rho-r-\frac{\lambda+\mu-\nu}{2}, \delta \right), (a_p, e_p) \\ \left(1-\rho-\frac{\lambda+\mu-\nu}{2}, \delta \right); \\ (b_q,\ f_q), \left(1-\rho-\frac{\lambda+\mu-\nu}{2}, \delta \right) \end{matrix} \right. \right], \quad (6)$$

where δ is a positive number and

$$A \leqslant 0,\ B > 0,\ |\arg z| < B\pi/2,$$
$$\operatorname{Re}(2\rho + 2\delta b_j/f_j) > 0,\quad j = 1, \ldots\ldots, m;$$
$$\operatorname{Re}(2\rho + 2\delta a_j/e_j) < 2\delta,\quad j = 1, 2, \ldots\ldots, n;$$
$$\operatorname{Re}(\lambda + \mu + \nu + 2\rho + 2\delta b_j/f_j) > 0,\quad j = 1, \ldots\ldots, m, 0 < a < b.$$

On applying the same procedure and using[3] [equation (7)], the following formula is obtained:

$$\int_0^a x^{2\rho-1}(a^2 - x^2)^{\sigma-1} I_\nu(x) \cdot H_{p,q}^{m,n}\left[z\, x^{2\delta}(a^2 - x^2)^{\delta} \,\middle|\, \begin{matrix}(a_p, e_p)\\ (b_q, f_q)\end{matrix}\right] . dx$$

$$= \frac{a^{\nu+2\rho+2\sigma-2}}{2^{\nu+1}} \cdot \sum_{r=0}^{\infty} \frac{(a^2/4)^r}{r!\,\Gamma(\nu+1+r)} \times$$

$$\times H_{p+2,q+1}^{m,n+2}\left[z.\, a^{4\delta} \,\middle|\, \begin{matrix}(1-\sigma, \delta), (1-\nu/2-\rho-r, \delta), (a_p, e_p)\\ (b_q, f_q), (1-\nu/2-\rho-\sigma-r, 2\delta)\end{matrix}\right], \tag{7}$$

where δ is a positive number and

$$A \leqslant 0,\ B > 0,\ |\arg z| < B.\pi/2,$$
$$\operatorname{Re}(2\rho + 2\delta b_j/f_j) > 0,\ \operatorname{Re}(\sigma + \delta b_j/f_j) > 0,\quad j = 1, \ldots\ldots, m;$$
$$\operatorname{Re}(\nu + \rho + \delta b_j/f_j) > 0,\quad j = 1, \ldots\ldots, m.$$

On applying the same procedure and using[3] equation (43) the following formula is obtained:

$$\int_0^\infty x^{\rho-1} \cdot e^{-x} \cos(4\, a\, x^{\frac{1}{2}})\, K_\nu(x) \cdot H_{p,q}^{m,n}\left[z.\, x^{\delta} \,\middle|\, \begin{matrix}(a_p, e_p)\\ (b_q, f_q)\end{matrix}\right] . dx$$

$$= (\pi)^{\frac{1}{2}} \cdot 2^{-\rho} \cdot \sum_{r=0}^{\infty} \frac{(-2a^2)^r}{r!\,(\frac{1}{2})_r} \times$$

$$\times H_{p+2,q+1}^{m,n+2}\left[z \cdot 2^{-\delta} \,\middle|\, \begin{matrix}(1-\rho-\nu-r, \delta), (1-\rho+\nu-r, \delta), (a_p, e_p)\\ (b_q, f_q), (\frac{1}{2}-\rho-r, \delta)\end{matrix}\right] \tag{8}$$

where δ is a positive number and

$$A \leqslant 0, B > 0, |\arg z| < B.\ \pi/2,$$
$$\operatorname{Re}(\rho + \delta b_j/f_j) > 0, j = 1, \ldots\ldots, m;$$
$$\operatorname{Re}(\rho + \delta a_j/e_j) < \delta, j = 1, \ldots\ldots, n,$$

On applying the same method and using[3] equation (50) the following formula is obtained:

$$\int_0^\infty x^{2\rho-1} \sin(2ax)\, K_\mu(x)\, K_\nu(x)\, H_{p,q}^{m,n}\left[z x^{2\delta} \,\middle|\, \begin{matrix}(a_p, e_p)\\ (b_q, f_q)\end{matrix}\right] dx$$

$$= (2\pi)^{\frac{1}{2}} \cdot 2^{-3/2} \cdot a \sum_{r=0}^{\infty} \frac{(-a2)^r}{r!\,(3/2)_r} \times$$

$$\times H_{p+4,q+2}^{m,n+4}\left[z \,\middle|\, \begin{matrix}(\frac{1}{2}-\rho-r-\mu/2-\nu/2, \delta), (\frac{1}{2}-\rho-r-\mu/2+\nu/2, \delta),\\ (\frac{1}{2}-\rho+\mu/2-\nu/2-r, \delta), (\frac{1}{2}-\rho-r+\mu/2+\nu/2, \delta), (a_p, e_p);\\ (b_q, f_q), (\frac{1}{2}-\rho-r, \delta), (-\rho-r, \delta)\end{matrix}\right], \tag{9}$$

where δ is a positive number and

$$A \leqslant 0\,,\ B > 0\,, |\arg z| < B\,.\,\pi/2\,,$$
$$\mathrm{Re}\,(2\rho + 2\delta\, b_j/f_j) > 0\,,\ j = 1\,,\ \ldots\ldots, m\,,$$
$$\mathrm{Re}\,(2\delta\ a_j/e_j + 2\rho) < 2\delta\,,\ j = 1\,,\ \ldots\ldots, n\,;$$
$$\mathrm{Re}\,|\alpha| < 1\,,$$
$$\mathrm{Re}\,(\rho + \delta\, b_j/f_j) > |\,\mathrm{Re}\,(\mu)\,| + |\ \mathrm{Re}\,\nu\,| - |\,,\ j = 1\,,\ \ldots\ldots, m.$$

On applying the same procedure and using[3] [equation (17)], the following formula is obtained

$$\int_0^\infty x^{2\rho-1}\ S_{\mu,\,\nu}(x).\ H^{m,\,n}_{p,\,q}\left[Z.\,x^{2\delta}\,\middle|\,\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right].\,dx$$

$$= 2^{2\rho+\mu-2}\,\Gamma\left(\frac{1+\mu+\nu}{2}\right)\,\Gamma\left(\frac{1+\mu-\nu}{2}\right)\times$$

$$\times\, H^{m+1,\,n+1}_{p+3,\,q+1}\left[z.\,2^{2\delta}\,\middle|\,\begin{matrix}(\tfrac{1}{2}-\mu/2-\rho,\,\delta),\ (a_p, e_p),\ \ (1-\nu/2-\rho,\,\delta),\,(1+\nu/2-\rho,\,\delta),\\(\tfrac{1}{2}-\mu/2-\rho,\,\delta),\ (b_q, f_q)\end{matrix}\right], \qquad (10)$$

where δ is a positive number and

$$A \leqslant 0,\ B > 0,\ |\arg z| < B.\ \pi/2\,,$$
$$\mathrm{Re}\ (2\rho + 2\delta\, b_j/f_j) > 0\,,$$
$$-\,\mathrm{Re}\,\mu < \mathrm{Re}\,(2\rho + 2\delta\, b_j/f_j) > 5/2\,,\ j = 1,\ \ldots\ldots, m\,;$$
$$\mathrm{Re}\,(2\rho + 2\delta\, a_j/e_j) < 2\delta\,,\ j = 1\,,\ \ldots\ldots, n.$$

PARTICULAR CASES

In (5), assuming δ as a positive integer, putting

$$e_j = f_i = 1\ (j = 1\,,\ \ldots\ldots, p\,;\, i = 1,\ \ldots\ldots, q)$$

using the formula

$$H^{m,\,n}_{p,\,q}\left[z\,\middle|\,\begin{matrix}(a_p,\ 1)\\(b_q\,,\ 1)\end{matrix}\right] = G^{m,\,n}_{p\,;\,q}\left[z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right],$$

and simplifying with the help of (1), and equations[4] (11) and (17), we obtain the following formula :

$$\int_0^a x^{2\rho-1}\ P_\mu(2x^2\alpha^{-2} - 1)\ J_\nu(x)\ G^{m,\,n}_{p\,,\,q}\left[zx^{2\delta}\ \middle|\ \begin{matrix}a_1,\ \ldots.\,,\ a_p\\b_1,\ \ldots.\,,\ b_q\end{matrix}\right]dx$$

$$= \frac{2^{-\nu-1}\ \alpha^{2\rho+\nu}\ (2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}\ \delta^{\rho-\mu+\nu/2-3/2}}{}\,.\sum_{r=0}^{\infty}\frac{(-\,\alpha^2/4)^r}{r\,!\,\Gamma\,(\nu+1+r)}\times$$

$$\times\, G^{m,\,n+3\delta}_{p+3\delta,\,q+2\delta}\left[z\delta^{\delta}\ \alpha^{2\delta}\,\middle|\,\begin{matrix}\triangle\,(\delta,\,1+\mu-\nu/2-\rho),\ \triangle\,(\delta,\,1-r-\nu/2-\rho),\\ \triangle\,(\delta,\,1-r-\nu/2-\rho),\ a_1,\ \ldots\ldots, a_p\,;\\ b_1,\ \ldots\ldots, b_q,\ \triangle\,(\delta,-\mu-r-\nu/2-\rho),\ \triangle\,(\delta,\,1+\mu-r-\nu/2-\rho)\end{matrix}\right], \qquad (11)$$

where δ is a positive number and

$$p + q < 2\ (m+n)\,,$$
$$|\arg z| < [m + n - (p+q)/2]\,.\,\pi\,,$$
$$\mathrm{Re}\,(\rho + \nu + \delta\, b_j) > 0,\ \mathrm{Re}\,(2\rho + 2\delta\, b_j) > 0\,,\ j = 1\,,\ \ldots\ldots, m\,;$$
$$\mathrm{Re}\,(2\rho + 2\delta a_j) < 2\delta\,,\ j = 1\,,\ldots\ldots, n.$$

Reducing (6) to G-function as above, we get the result

$$\int_0^\infty x^{2\rho-1}\; J_\lambda(ax)\; J_\mu(ax)\; J_\nu(2bx)\; G\,{}^{m,\,n}_{p,\,q}\left[zx^{2\delta}\;\middle|\;\begin{matrix} a_1, & \ldots\ldots, & a_p \\ b_1, & \ldots\ldots, & b_q\end{matrix}\right].\,dx$$

$$= a^{\lambda+\mu}\, b^{-\lambda-\mu-2\rho}.\, 2^{-\lambda-\mu}.\, \delta^{\lambda+\mu-1+2\rho} \times$$

$$\times \sum_{r=0}^{\infty} \frac{\delta^{2r}\left(\lambda+\mu+\frac{1}{2}\right)_r \left(\frac{\lambda+\mu}{2}+1\right)_r \left(\frac{a^2}{b^2}\right)^r}{\Gamma(\lambda+1+r)\,\Gamma(\mu+1+r)\,(\lambda+\mu+1)_r\, r\,!} \times$$

$$\times\; G\,{}^{m,\,n+2\delta}_{p+3\delta,\;q+\delta}\left[z\,(\delta/b)^{2\delta}\;\middle|\;\begin{matrix}\triangle\left(\delta,1-\rho-r-\frac{\lambda+\mu+\nu}{2}\right),\triangle\left(\delta,1-\rho-r-\frac{\lambda+\mu-\nu}{2}\right),\\ a_1,\ldots\ldots,a_p,\triangle\left(\delta,1-\rho-\frac{\lambda+\mu-\nu}{2}\right);\\ b_1,\ldots\ldots,b_q,\triangle\left(\delta,1-\rho-\frac{\lambda+\mu-\nu}{2}\right)\end{matrix}\right], \qquad (12)$$

where δ is a positive integer and

$$p+q<2\,(m+n)\,,$$
$$|\arg z|<[m+n-(p+q)/2].\,\pi\,,$$
$$\operatorname{Re}\,(2\rho+2\delta b_j)>0\,,\; j=1,\;\ldots\ldots\; m\,;$$
$$\operatorname{Re}\,(2\rho+2\delta a_j)<2\delta\,,\; j=1,\;\ldots\ldots,n\,;$$
$$\operatorname{Re}\,(\rho+\mu+\nu+2\rho+2\delta\, b_j)>0\,,\; j=1,\;\ldots\ldots,m\,;\,0<a<b\,.$$

Reducing (7) to G-function as above, we have

$$\int_0^{\alpha} x^{2\rho-1}\,(\alpha^2-x^2)^{\sigma-1}\, I_\nu(x)\; G\,{}^{m,\,n}_{p,\,q}\left[zx\quad(\alpha^2-x^2)^{\delta}\;\middle|\;\begin{matrix}a_1, & \ldots\ldots, & a_p\\ b_1, & \ldots\ldots, & b_q\end{matrix}\right]dx$$

$$=\frac{\alpha^{\nu+2\rho+2\sigma-2}.\,\delta^{-\frac{1}{2}}}{2^{3/2\nu+\rho+\sigma+1/2}}\cdot\sum_{r=0}^{\infty}\frac{2^{-r}\,(\alpha^2/4)^r}{\Gamma(\nu+1+r)r\,!}\times$$

$$\times\; G\,{}^{m,\,n+2\delta}_{p+2\delta,\,q+2\delta}\left[\frac{z.\,\alpha^{4\delta}}{2^{2\delta}}\;\middle|\;\begin{matrix}\triangle\,(\delta,1-\sigma),\triangle\,(\delta,1-\nu/2-\rho-r),a_1,\ldots\ldots,a_p\,;\\ b_1,\ldots\ldots,b_q,\triangle\,(2\delta,1-\nu/2-\rho-\sigma-r)\end{matrix}\right], \qquad (13)$$

where δ is a positive integer and

$$p+q<2\,(m+n)\,,$$
$$|\arg z|<[m+n-(p+q)/2].\,\pi,$$
$$\operatorname{Re}\,(2\rho+2\delta\, b_j)>0,\;\; \operatorname{Re}\,(\sigma+\delta b_j)>0,\;\; j=1,\ldots\ldots,m\,;$$
$$\operatorname{Re}\,(\nu+\rho+\delta b_j)>0.$$

Reducing (8) to G-function as above, we have the following result :

$$\int_0^\infty x^{\rho-1}.\, e^{-x}.\,\cos\,(4\alpha\, x^{1/2})\; K_\nu(x)\; G\,{}^{m,\,n}_{p,\,q}\left[zx^{\delta}\;\middle|\;\begin{matrix}a_1, & \ldots\ldots, & a_p\\ b_1, & \ldots\ldots, & b_q\end{matrix}\right].\,dx$$

$$=(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}.\,(\pi)^{1/2}.\,2^{-\rho}.\,\delta^{\rho-1}.\sum_{r=0}^{\infty}\frac{\delta^{r}.\,(-2\,\alpha^2)_r}{r\,!\,(1/2)_r}\times$$

$$\times\; G\,{}^{m,\,n+2\delta}_{p+2\delta,\,q+\delta}\left[z\quad(\delta/2)^{\delta}\;\middle|\;\begin{matrix}\triangle(\delta,1-\rho-\nu-r),\triangle\,(\delta,1-\rho+\nu-r),a_1,\ldots\ldots,a_p\,;\\ b_1,\ldots\ldots,b_q,\triangle\,(\delta,\frac{1}{2}-\rho-r)\end{matrix}\right], \qquad (14)$$

where δ is a positive integer and

$$p+q<2(m+n),$$
$$|\arg z|<[m+n-(p+q)/2]\,.\,\pi,$$
$$\operatorname{Re}(\rho+\delta b_j)>0;\quad j=1,\ldots\ldots,m;$$
$$\operatorname{Re}(\rho+\delta a_j)<\delta,\quad j=1,\ldots\ldots,n.$$

Reducing (9) to G-function, we have the formula

$$\int_0^\infty x^{2\rho-1}\sin(2ax)\,K_\mu(x)\,K_\nu(x)\,G^{m,n}_{p,q}\left[z.\,x^{2\delta}\left|\begin{matrix}a_1,\ldots\ldots,a_p\\ b_1,\ldots\ldots,b_q\end{matrix}\right.\right].dx$$

$$=(2\pi)^{3/2-\delta}.\,2^{-3/2}.\,a.\,\delta^{2\rho-1/2}.\sum_{r=0}^{\infty}\frac{\delta^{2r}(-a^2)^r}{r!\,(1/2)_r}\times$$

$$\times\ G^{m,\,n+4\delta}_{p+4\delta,\,q+2\delta}\left[z.\,\delta^2\left|\begin{matrix}\triangle\left(\delta,1-\rho-r-\frac{\mu+\nu+1}{2}\right),\triangle\left(\delta,1-\rho-r-\frac{\mu-\nu+1}{2}\right),\\ \triangle\left(\delta,1-\rho-r-\frac{\nu-\mu+1}{2}\right),\triangle\left(\delta,1-\rho-r-\frac{1-\mu+\nu}{2}\right),\\ a_1,\ldots\ldots,a_p;\\ b_1,\ldots\ldots,b_q,\triangle(\delta,-\rho-r),\triangle(\delta,\tfrac{1}{2}-\rho-r)\end{matrix}\right.\right],\qquad(15)$$

where δ is a positive integer and

$$p+q<(m+n),$$
$$|\arg z|<[m+n-(p+q)/2].\,\pi,$$
$$\operatorname{Re}(2\rho+2\delta\,b_j)>0,\ j=1,\ldots\ldots,m;$$
$$\operatorname{Re}(2\rho+2\delta a_j)<2\delta,\ j=1,\ \ldots\ldots,n;$$
$$|\operatorname{Re}a|<1,$$
$$\operatorname{Re}(\rho+\delta\,b_j)>|\operatorname{Re}\mu|+|\operatorname{Re}\nu|-1,\ j=1,\ \ldots\ldots\ldots,m.$$

Reducing (10) to G-function as above, we have the result

$$\int_0^\infty x^{2\rho-1}.\,S_{\mu,\,\nu}(x)\,G^{m,n}_{p,q}\left[z.\ x^{2\delta}\left|\begin{matrix}a_1,\ldots\ldots,a_p\\ b_1,\ldots\ldots,b_q\end{matrix}\right.\right].dx$$

$$=\delta^{2\rho-1}.\,2^{2\rho+\mu-2}\,\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)\times$$

$$\times\,G^{m+\delta,\,n+\delta}_{p+3\delta,\,q+\delta}\left[z\,(2\delta)^{2\delta}\left|\begin{matrix}\triangle\left(\delta,\tfrac{1}{2}-\frac{\mu}{2}-\rho\right),a_1,\ldots\ldots,a_p,\triangle\,\delta,1+\nu/2-\rho),\\ \triangle(\delta,1-\nu/2-\rho);\\ \triangle\left(\delta,\frac{1-\mu}{2}-\rho\right),b_1,\ldots\ldots,b_q\end{matrix}\right.\right],\qquad(16)$$

where δ is a positive integer and

$$p+q<2(m+n),$$
$$|\arg z|<[m+n-(p+q)/2].\,\pi,$$
$$\operatorname{Re}(2\rho+2\delta\,b_j)>0,$$
$$-\operatorname{Re}\mu<\operatorname{Re}(2\rho+2\delta b_j)<5/2,\ j=1,\ \ldots\ldots,m;$$
$$\operatorname{Re}(2\rho+2\delta\,a_j)<2\delta,\ j=1,\ \ldots\ldots,n.$$

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