OSCILLATIONS OF SECOND-ORDER FLUIDS NEAR A SPHERE

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The high frequency oscillations of second-order fluids near a fixed sphere have been considered. The peculiarity of the second-order fluid is that the magnitude of the steady secondary streaming at large distance from the sphere depends upon all the material constants, while for a Newtonian fluid, this is not so. The drag on the sphere has been calculated for various values of the non-dimensional parameters formed out of the material constants and the flow parameters. The outer motion is pushed away from the sphere with increase in the non-dimensional parameters.

The small oscillations of a body in a Newtonian fluid at rest induce characteristic secondary flow whose nature is such that a steady motion is imparted to the whole fluid inspite of the fact that the motion of the body is purely periodic. This was first pointed out by Reyleigh[1] in his work on Kundt's dust tubes. Schlichting[2] has discussed the flow induced by a cylinder performing high frequency oscillations along an axis perpendicular to its generators in a viscous fluid. He has shown that when $\epsilon = (U_\infty / d\omega) << 1$, the first order fluctuating flow is confined to a 'shear wave' region of thickness $O\,(\nu_1/\omega)^{\frac{1}{2}}$, beyond which a steady flow $O\,(\epsilon\, U_\infty)$ persists and is characterised by $\epsilon\, R = U_\infty^2 / \omega\ \nu_1$. Here ν_1 is kinematic viscosity. ω the frequency of oscillations, U_∞ and d are respectively a typical velocity and length. He has used the Blasius boundary layer equation for plane motion thus neglecting the effect of the curvature. Stuart[3], Kelley[4] and Riley[5] have attempted such problems. Recently Wang[6,7] has derived the boundary layer equation for cylindrical surface in another manner and showed that the interaction of the curvature and the velocity is important in second approximation in ϵ.

The boundary layer equation for an axisymmetric flow of second-order fluids past a sphere have been derived following Wang[7]. Using this equation, we have discussed the small oscillations of a stream of the fluid along an axis of symmetry of a fixed sphere in it. The flow so caused is equivalent to that induced by a sphere oscillating along its axis of symmetry in the fluid at rest. Assuming that the amplitude of oscillations is very small compared with the radius of the sphere, we have obtained first and second approximations in $\epsilon = U_\infty / \omega\, d$ where d is the radius of the sphere, U_∞ the velocity at infinity and ω the frequency of oscillations. The magnitude of the steady streaming at infinity and the drag on the sphere have been calculated for various values of the non-dimensional parameters formed out of the material constants and ω. The typical flow pattern has been shown in Fig. 1.

BOUNDARY LAYER EQUATION

The constitutive equation of an incompressible second-order fluid has been suggested by Coleman & Noll[8] as

$$S^i{}_j = -\,p\,\delta^i{}_j + \mu_1\,A^i{}_j + \mu_2\,B^i{}_j + \mu_3\,A^i{}_k\,A^k{}_j \quad (1)$$

where

$$A^i{}_j = v^i{}_{;\,j} + g^{ik}\,v_{j;\,k}\,,\ B^i{}_j = a^i{}_{;\,j} + g^{ik}\,a_{j;\,k} + 2\,g^{il}\,v_{k;\,l}\,v^k{}_{;\,j}. \quad (2)$$

$S^i{}_j$ is a stress tensor; p is an undetermined hydrostatic pressure ; μ_1, μ_2 and μ_3 are material constants; v^i and a^i are respectively velocity and acceleration vectors, and a semi-colon denotes covariant differentiation with respect to the symbol following it. g^{ij} is a conjugate metric tensor.

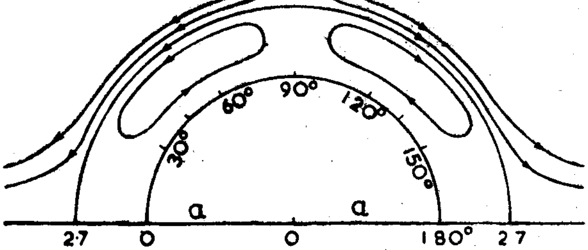

Fig. 1—The typical flow pattern in the first and second quadrants for $(\alpha, \beta) = (-0\cdot1, 0\cdot5)$.

The Cauchy's equation of motion and the equation of continuity are

$$\rho a^i = S^{ij}_{\ ;j}$$
$$v^i_{\ ,i} = 0 \tag{4}$$

where ρ is the density of the fluid.

We consider a stream of a second-order fluid, which oscillates with frequency ω along an axis of symmetry of a fixed sphere of radius d within it. We introduce a system of spherical polar coordinates (r, θ, ϕ) with the origin at the centre of the sphere and $\theta = 0$ as the axis of symmetry. Let v_r, v_θ and v_ϕ denote the velocity components in the directions of r, θ and ϕ respectively. The potential flow outside the boundary layer region is given by

$$V(\theta, t) = \frac{3}{2} U_\infty \sin\theta \cos\omega t \tag{5}$$

The boundary conditions of this problem are

$$r = d: v_r = 0, v_\theta = 0,$$
$$r = d + \delta: v_r = 0, v_\theta = V(\theta, t) \tag{6}$$

Here δ is the boundary layer thickness. There is no flow in the direction of ϕ and all entities are independent of ϕ; hence the flow is axisymmetric.

We discuss the case when the amplitude of oscillations is very small compared with the radius of the sphere, i.e. $\epsilon = (U_\infty / d\omega) << 1$. Assuming a thin boundary layer of order ϵ, we introduce

$$u = \frac{1}{\epsilon U_\infty} v_r, \; v = \frac{1}{U_\infty} v_\theta, \; y = \frac{1}{\epsilon}\left(\frac{r}{d} - 1\right)$$
$$\tau = \omega t, \; p_1 = \frac{1}{\rho\omega d U_\infty} p, \; \delta_1 = \frac{1}{\epsilon d}\delta. \tag{7}$$

All the non-dimensional entities introduced in (7) are of order unity. Substituting (1) and (2), (3) yields the following non-dimensional equation in the directions of y and θ respectively.

$$\frac{\partial}{\partial y}\left\{-p_1 + \epsilon\sigma(2\alpha + \beta)\left(\frac{\partial v}{\partial y}\right)^2\right\} + O(\epsilon^2) = 0 \tag{8}$$

$$\frac{\partial v}{\partial \tau} + \epsilon\left(u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial \theta}\right) = -\frac{1}{1 + \epsilon y}\frac{\partial}{\partial\theta}\left\{p_1 - \epsilon\sigma(2\alpha+\beta)\left(\frac{\partial v}{\partial y}\right)^2\right\} +$$
$$+ \sigma\left(1 + \alpha\frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2 v}{\partial y^2} + 2\epsilon\frac{\partial v}{\partial y}\right) - \epsilon\sigma\beta\left\{2v\frac{\partial^2 v}{\partial y^2} +\right.$$
$$\left. + \left(\frac{\partial v}{\partial y}\right)^2\right\}\cot\theta + \epsilon\sigma\alpha\left\{\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} + u\frac{\partial^3 v}{\partial y^3} + v\frac{\partial^3 v}{\partial\theta\,\partial y^2} +\right.$$
$$\left. + \frac{\partial v}{\partial\theta}\frac{\partial^2 v}{\partial y^2} - 2v\frac{\partial^2 v}{\partial y^2}\cot\theta\right\} + O(\epsilon^2) \tag{9}$$

where $\sigma = \mu_1\omega/\rho U_\infty^2$, $\alpha = \mu_2\omega/\mu_1$ and $\beta = \mu_3\omega/\mu_1$.

These non-dimensional parameters have been assumed to be of order unity.

Equation (8) shows that within terms $O(\epsilon)$, the modified pressure $p_1 - \epsilon\sigma(2\alpha + \beta)(\partial v/\partial y)^2$ does not vary across the boundary layer, i.e. it is same within and outside the boundary layer region. If π denotes the pressure outside the boundary layer region, then

$$\pi = p_1 - \epsilon\sigma(2\alpha + \beta)(\partial v/\partial y)^2. \tag{10}$$

The pressure π is governed by

$$\frac{1}{U_\infty}\frac{\partial V}{\partial \tau}+\frac{\epsilon}{U_\infty^{2}}V\frac{\partial V}{\partial \theta}=-\frac{1}{1+\epsilon y}\frac{\partial \pi}{\partial \theta} \tag{11}$$

Neglecting the terms $O(\epsilon^2)$ in (9) and using (10) and (11), we obtain the boundary layer equation as:

$$\frac{\partial v}{\partial \tau}+\epsilon\left(u\frac{\partial v}{\partial y}+v\frac{\partial v}{\partial \theta}\right)=\frac{1}{U_\infty}\frac{\partial V}{\partial \tau}+\frac{\epsilon}{U_\infty^{2}}V\frac{\partial V}{\partial \theta}+\sigma\left(1+\alpha\frac{\partial}{\partial \tau}\right)\left(\frac{\partial^2 v}{\partial y^2}+2\epsilon\frac{\partial v}{\partial y}\right)-$$
$$-\epsilon\sigma\beta\left\{2v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right\}\cot\theta+\epsilon\sigma\alpha\left\{\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}+u\frac{\partial^3 v}{\partial y^3}+\right.$$
$$\left.+v\frac{\partial^3 v}{\partial y^2\partial\theta}+\frac{\partial v}{\partial \theta}\frac{\partial^2 v}{\partial y^2}-2v\frac{\partial^2 v}{\partial y^2}\cot\theta\right\}. \tag{12}$$

The equation of continuity (4) in non-dimensional form is

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial \theta}+v\cot\theta+\epsilon\left(2u-y\frac{\partial u}{\partial y}\right)=0 \tag{13}$$

The boundary conditions (6) can now be written as

$$y=0:\ u=0\,,\ v=0\,,$$
$$y=\delta_1:\ u=0\,,\ v=\frac{3}{2}\sin\theta\, e^{i\tau} \tag{14}$$

The complex notation has been used here with the convention that only the real part of a complex quantity has a physical significance

SOLUTION OF EQUATION

To obtain an approximate solution of the boundary layer equation (12), we expand u and v in powers of ϵ as follows:

$$u=u_0+\epsilon u_1+O(\epsilon^2)\,,\ v=v_0+\epsilon v_1+O(\epsilon^2) \tag{15}$$

The boundary conditions (14) become

$$y=0:\ u_0=v_0=0\,,\ u_1=v_1=0\,,$$
$$y=\delta_1:\ u_0=0,\ v_0=\frac{3}{2}e^{i\tau}\sin\theta\,,\ u_1=v_1=0\,. \tag{16}$$

Substituting u and v from (15) into (12) and (13) and comparing the coefficients of like powers of ϵ on both sides, we get

$$\left.\begin{aligned}&\text{(a)}\ \sigma\left(1+\alpha\frac{\partial}{\partial\tau}\right)\frac{\partial^2 v_0}{\partial y^2}-\frac{\partial v_0}{\partial \tau}=-\frac{3}{2}\,i\sin\theta\, e^{i\tau}\,,\\&\text{(b)}\ \frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial \theta}+v_0\cot\theta=0\,;\end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned}&\text{(a)}\ \alpha\left(1+\alpha\frac{\partial}{\partial\tau}\right)\frac{\partial^2 v_1}{\partial y^2}-\frac{\partial v_1}{\partial \tau}=u_0\frac{\partial v_0}{\partial y}+v_0\frac{\partial v_0}{\partial \theta}-\frac{9}{16}(1+e^{2i\tau})\sin 2\theta-\\&\quad-2\sigma\left(1+\alpha\frac{\partial}{\partial\tau}\right)\frac{\partial v_0}{\partial y}-\sigma\alpha\left\{u_0\frac{\partial^3 v_0}{\partial y^3}+\frac{\partial v_0}{\partial y}\frac{\partial^2 u_0}{\partial y^2}+\right.\\&\quad\left.+v_0\frac{\partial^3 v_0}{\partial y^2\partial\theta}+\frac{\partial v_0}{\partial\theta}\frac{\partial^2 v_0}{\partial y^2}-2v_0\frac{\partial^2 v_0}{\partial y^2}\cot\theta\right\}+\\&\quad+\sigma\beta\left\{2v_0\frac{\partial^2 v_0}{\partial y^2}+\left(\frac{\partial v_0}{\partial y}\right)^2\right\}\cot\theta\,,\\&\text{(b)}\ \frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial\theta}+v_1\cot\theta+2u_0-y\frac{\partial u_0}{\partial y}=0\,.\end{aligned}\right\} \tag{18}$$

Solving (17) under the conditions (16), we get the first order solution as

$$v_0 = \frac{3}{2}\left\{\cos\tau - e^{-A\eta}\cos(\tau - B\eta)\right\}\sin\theta ,$$

$$u_0 = -3(\sigma/AB)^{\frac{1}{2}}\left\{2^{\frac{1}{2}}\eta\cos\tau - \cos(\tau-\psi) + e^{-A\eta}\cos(\tau - B\eta - \psi)\right\}\cos\theta . \quad (19)$$

where $\eta = (AB/2\sigma)^{\frac{1}{2}}\, y$,

$$A = 2^{\frac{1}{2}}\cos\psi = \left(1 + \frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

$$B = 2^{\frac{1}{2}}\sin\psi = \left(1 - \frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

The condition that $u_0 \to 0$ as $\eta \to \infty$ cannot be satisfied and hence the solution is valid only near the surface of the sphere.

We use the solution (19) to obtain the second approximations u_1 and v_1 from (18). It is noted that the convective part of the inertia terms and the non-Newtonian terms in (18 a) will contribute terms with $\cos^2\tau$, which in turn will give terms with $\cos 2\tau$, $\sin 2\tau$ and steady state terms, i.e. time independent terms. The viscous terms will contribute terms with $\cos\tau$ and $\sin\tau$. Taking this fact into account, we take

$$v_1(\eta, \theta, \tau) = \frac{9}{8}\left\{f(\eta) + g(\eta)\, e^{2i\tau}\right\}\sin 2\theta + \frac{3}{2}\left(\frac{2\sigma}{AB}\right)^{\frac{1}{2}} h(\eta)\, e^{i\tau}\sin\theta . \quad (20)$$

The conditions on f, g and h are

$$f = g = h = 0 \text{ at } \eta = 0 \text{ and } \eta = \left(\frac{AB}{2\sigma}\right)^{\frac{1}{2}}\delta_1 \quad (21)$$

Substituting the expressions for u_0 and v_0 from (19) and that for v_1 from (20) into (18 a), and equating the co-efficients of like powers of $e^{i\tau}$ on both sides, we get a set of ordinary differential equations determining f, g and h. Solving them under conditions (21), we get f and h as follows :

$$f(\eta) = \left\{\frac{3A^2 - 4}{AB} + \alpha(A^2 - 3) + \beta\right\} + \frac{1}{4A^2}\left(e^{-2A\eta} - 1\right)\left\{\frac{3 - 2A^2}{AB} + 4\alpha(A^2-1)^2 + \right.$$
$$\left. + \beta(2A^2 - 1)\right\} + e^{-A\eta}\left\{\left(\frac{1}{AB} + 2\alpha - \beta\right)\cos B\eta + \frac{3}{AB}\cos(B\eta + 2\psi) - \right.$$
$$\left. - \alpha\cos(B\eta - 2\psi) - 2^{\frac{1}{2}}\eta\left\{\frac{1}{AB}\cos(B\eta + \psi) - \alpha\cos(B\eta - \psi)\right\}\right\} ,$$

$$h(\eta) = \eta\, e^{-A\eta}\cos B\eta . \quad (22)$$

$g(\eta)$ can also be obtained in a similar manner.

Regarding the steady state contribution $f(\eta)$, it is found that only the boundary condition at the wall can be satisfied, and that at a large distance from it, it is possible to make the tangential component finite but not zero. Thus

$$f(\infty) = \frac{1}{4A^3B}(12A^4 - 14A^2 - 3) - \frac{\alpha}{A^2}(1 + A^2) + \frac{\beta}{4A^2}(1 + 2A^2) .$$

The second approximation is seen to contain a steady state term which does not vanish at a large distance from the surface of the sphere, i.e. outside the boundary layer region. Its magnitude is given by

$$v_1(\infty, \theta) = \frac{9}{8} f(\infty)\sin 2\theta . \quad (23)$$

The values of $f(\infty)$ have been calculated for $(\alpha, \beta) = (0, 0)$, $(-0\cdot1, 0\cdot5)$, $(-0\cdot2, 1\cdot0)$, $(-0\cdot3, 1\cdot2)$ and $(-0\cdot4, 1\cdot6)$, and been given in Table 1.

We proceed to study the secondary steady streaming in detail. We introduce the stream function Ψ corresponding to this flow in the following form

$$\Psi = \frac{9}{4}\left(2\,\sigma/AB\right)^{\frac{1}{2}} \sin\theta \sin 2\theta \int_0^{\eta} f(\eta)\, d\eta = 9\left(2\,\sigma/AB\right)^{\frac{1}{2}} \sin^2\theta \cos\theta\, F(\eta) \qquad (24)$$

where

$$F(\eta) = \frac{1}{2}\,\eta f(\infty) + \frac{1}{16A^3}\left(1 - e^{-2A\eta}\right)\left\{\frac{3-2A^2}{AB} + 4\alpha\left(A^2-1\right)^2 + \beta(2A^2-1)\right\} + + \frac{A}{4}\left\{\frac{13-8A^2}{AB} + 2\alpha - \beta\right\} - \frac{1}{2}\,\eta\, e^{-A\eta}\left\{\frac{1}{AB}\cos(B\eta + 2\psi) - \alpha\cos B\eta\right\} - - 8^{-\frac{1}{2}}\, e^{-A\eta}\left\{\left(\frac{1}{AB} + 3\alpha - \beta\right)\cos(B\eta+\psi) - \alpha\cos(B\eta - \psi) - - \frac{1}{AB}\cos(B\eta + 3\psi)\right\}. \qquad (25)$$

The corresponding expression for $F(\eta)$ for Newtonian fluid can be obtained from (25) by setting $\alpha = \beta = 0$ and $A = B = 1$. Thus $F_N(\eta)$ for Newtonian fluid is

$$F_N(\eta) = \frac{21}{16} - \frac{5}{8}\,\eta - \frac{1}{16}\,e^{-2\eta} - \frac{1}{4}\,e^{-\eta}\,(2\eta\sin\eta + 5\cos\eta + 3\sin\eta) \qquad (26)$$

This is same as expression (32) in Riley[5].

The positive zero of $F(\eta)$ corresponds to a stream line which separates two circulatory motions in each quadrant. The positive zeros of $F(\eta)$ have been calculated for $(\alpha, \beta) = (0, 0)$, $(-0\cdot1, 0\cdot5)$, $(-0\cdot2, 1\cdot0)$ and $(-0\cdot3, 1\cdot2)$. They are respectively at $\eta = 1\cdot628$, $2\cdot7$, $4\cdot0$ and $4\cdot5$. The flow pattern in first two quadrants has been shown in Fig. 1 for $(\alpha, \beta) = (-0\cdot1, 0\cdot5)$.

The flow pattern for other values of α, β is similar to this except that the region of inner closed circulatory motion in each quadrant is expanded and outer motion is pushed away from the sphere with increase in values of α, β.

DRAG ON SPHERE

The drag D on the sphere is given by

$$D = \frac{1}{\pi d^3 \rho \omega U_\infty}\int_0^{\pi}\left[S_{rr}\cos\theta - S_{r\theta}\sin\theta\right]_{r=d} 2\pi d^2 \sin\theta\, d\theta$$

$$= -2\int_0^{\pi}\left[\Pi\cos\theta + \epsilon\,\sigma\left(1 + \alpha\frac{\partial}{\partial\tau}\right)\left(\frac{\partial v}{\partial y}\right)_{y=0}\sin\theta\right]\sin\theta\, d\theta \qquad (27)$$

TABLE 1

VALUES OF $f(\infty)$ FOR DIFFERENT VALUES OF (α, β)

(α, β)	$(0\cdot0)$	$(-0\cdot1, 0\cdot5)$	$(-0\cdot2, 1\cdot0)$	$(-0\cdot3, 1\cdot2)$	$(-0\cdot4, 1\cdot6)$
$f(\infty)$	$-1\cdot2500$	$-1\cdot0396$	$-0\cdot8016$	$-0\cdot7788$	$-0\cdot5509$
α	0	$-0\cdot1$	$-0\cdot2$	$-0\cdot3$	$-0\cdot4$
$\lvert D\rvert$	$2\cdot028$	$2\cdot027$	$2\cdot025$	$2\cdot024$	$2\cdot023$

Performing necessary integration, we obtain D within terms $O(\epsilon)$ in the following form

$$D = -2\{i + \epsilon(1 + i\alpha)(A + iB)(2\sigma AB)^{\frac{1}{2}}\}e^{i\tau} . \qquad (28)$$

We note that the drag D on the sphere is purely oscillatory in character. The amplitude of its oscillations is given by

$$|D| = 2[1 + 2\epsilon(\alpha A + B)(2\sigma AB)^{\frac{1}{2}}]^{\frac{1}{2}} \qquad (29)$$

The values of $|D|$ have been calculated for various values of α when $\sigma = 1$, $\epsilon = 0{\cdot}01$ and have been given in Table 1.

DISCUSSION

The corresponding results for the Newtonian fluid can be deduced from the above results by setting $\alpha = \beta = 0$.

A potential flow which is periodic with respect to time induces a steady secondary motion at a large distance from the surface of the sphere. The magnitude of this steady streaming decreases with increase in non-Newtonian parameters α and β. The peculiarity of the second-order fluid is that the magnitude of the steady streaming depends upon all the material constant though it is independent of viscosity in Newtonian fluid. This peculiarity has been noted by Srivastava & Saroa[9] in that the location of the point of separation for flow of a second-order fluid past a circular cylinder depends upon all the material constants, though for Newtonian fluid, it is not so.

The drag on the sphere is purely oscillatory in character. The amplitude of its oscillations decreases with increase in parameter but it is independent of β (and so of μ_3). The typical flow pattern in Fig. 1 shows that, in each quadrant, there are two circulatory motions separated by a stream line. The size of the inner region where the stream lines are closed increases and the outer motion is pushed away from the sphere with increase in the parameters.

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