INTERNAL CONICAL FLOW ABOUT UNYAWED CONES AT HIGH SPEEDS

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An approximate closed form solution to the internal, conical, inviscid, hypersonic flow about a circular cone at zero angle of incidence, has been obtained based on assumption that dissociation, and ionization etc. do not occur behind the shock wave. It has been shown that a singular line exists beyond which the solution is not valid.

The numerical solution of steady, inviscid, supersonic or hypersonic flow over a circular cone at zero angle of incidence has been obtained by Taylor & Maccoll[1]. An approximate closed form solution of the above problem has been obtained by Pottsepp[2]. Zienkiewics[3] has obtained the closed form solution in the presence of dissociation. They have considered that the flow downstream of the shock is irrotational and there is no characteristic length involved in the problem and hence all flow properties behind the shock depend upon θ only. But they have not discussed the occurrence of singularity which limits the downstream extent of the flow. Mölder[4] has discussed the existence of singular line in the case of internal, axisymmetric, hypersonic, conical flow past a circular cone by numerically solving the Taylor-Maccoll equation.

Following Pottsepp[2], we have obtained an approximate closed form solution which is valid only in the region between the shock and the singular line. The present investigation provides an analytical method for calculating an axisymmetric duct or inlet leading edge shape for a given leading edge shock.

BASIC EQUATIONS

We consider an infinite solid circular cone set at zero angle of attack in a uniform inviscid supersonic or hypersonic calorically and thermally perfect gas. It is well known that if the cone angle is less or the free-stream Mach number is greater than a certain critical value, an attached straight shock wave springs from the vertex[1]. If we assume that viscosity is zero and thermodynamic equilibrium exists, then all the flow properties downstream of the shock wave depend only on the angular variable θ, there being no characteristic length in the problem[1,3]. Let us assume a spherical polar co-ordinate system at O with θ measured from the free-stream direction and r measured radially from O as shown in Fig. 1. At the shock cone, $\theta = \psi$, the angle between the shock cone and the free-stream (the shock angle) is the same at all points on the cone surface[4].

Under these conditions, the equation of continuity is:

$$\frac{d}{d\theta}(\rho\, v \sin\theta) + 2\,\rho\, u \sin\theta = 0 \tag{1}$$

and the momentum equations in the r and θ directions are :

$$\frac{du}{d\theta} - v = 0 \tag{2}$$

$$v\,\frac{dv}{d\theta} + vu + \frac{1}{\rho}\,\frac{dp}{d\theta} = 0 \tag{3}$$

where u, v denote velocity components along r and θ directions, ρ is the density and p is the pressure. Equation (2) implies that the flow downstream of the shock is irrotational and therefore homentropic[3]. Let us define the velocity of sound, a, by the relation $(\partial p/\partial \rho)_s = a^2$, where the suffix s, indicates that the derivative is taken at constant entropy and thermal equilibrium. Although the sound waves may propagate at somewhat different speeds, depending upon their frequencies, the above definition of sound is consistant with the assumption of zero viscosity and relaxation time[3]. The magnitude of the free-stream Mach number is taken such that the temperature behind the shock wave is not so high that vibration, dissociation etc. might appear. Hence the fluid can be assumed to be a perfect gas with constant specific heats. From equations (1) to (3), we get[2,3] :

$$\left(1-\frac{v^2}{a^2}\right)\frac{d^2u}{d\theta^2}+\cot\theta\,\frac{du}{d\theta}+\left(2-\frac{v^2}{a^2}\right)u=0 \tag{4}$$

For a high free-stream Mach number, which is normally encountered in hypersonic flow, the shock layer is thin and v is small compared to a. Therefore it is reasonable to neglect (v^2/a^2) compared[2] to 1. Under this assumption, (4) reduces[2] to

$$\frac{d^2u}{d\theta^2}+\cot\theta\,\frac{du}{d\theta}+2u=0 \tag{5}$$

The boundary conditions are obtained from oblique shock[2] relations :

At
$$\theta=\psi : u = q_\infty \cos\psi$$
$$v=\frac{du}{d\theta}$$
$$=-q_\infty\,\frac{(K-1)\,M_\infty{}^2\sin^2\psi+2}{(K+1)\,M_\infty{}^2\sin^2\psi}\,\sin\psi \tag{6}$$

where q_∞, M_∞ and K are respectively free-stream velocity, free-stream Mach number, and specific heat ratio.

SOLUTIONS OF EQUATIONS

For a given semi-vertical angle of the cone, there is a critical free-stream Mach number for which the shock waves are attached[3]. The relation between the free-stream Mach number (M_∞), the semi-vertical angle of the cone (α), and the shock wave angle (ψ_1)—which is measured from the axis of the cone as shown in Fig. 1, can be expressed[2] as :

$$M_\infty=\operatorname{cosec}\psi_1\left[\frac{2\left\{B+ln\left(\dfrac{\sin\psi_1}{1-\cos\psi_1}\right)-\sec\psi_1\right\}}{\dfrac{(K+\cos 2\psi_1)}{\sin^2\psi_1\cos\psi_1}+2\left\{B+ln\left(\dfrac{\sin\psi_1}{1-\cos\psi_1}\right)\right\}}\right]^{\frac{1}{2}} \tag{7}$$

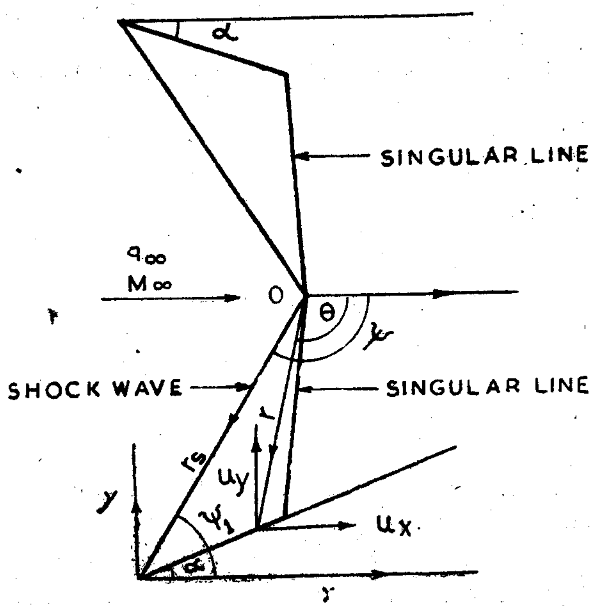

Fig. 1—Internal conical flow.

where

$$B=-\,ln\left(\frac{\sin\alpha}{1-\cos\alpha}\right)-\frac{\cos\alpha}{\sin^2\alpha}$$

The relation between ψ and ψ_1 is given by

$$\psi=\pi-\psi_1 \text{ (see Fig. 1)}$$

After the determination of the shock wave angle for a given free-stream Mach number and semi-vertical angle of the cone, we shall obtain the solution of (5) under the boundary conditions given by (6):

$$u=A_1\cos\theta\left[B_1+ln\left(\frac{\sin\theta}{1-\cos\theta}\right)-\sec\theta\right] \tag{8}$$

$$v=\frac{du}{d\theta}=-A_1\sin\theta\left[B_1+ln\left(\frac{\sin\theta}{1-\cos\theta}\right)+\frac{\cos\theta}{\sin^2\theta}\right] \tag{9}$$

where

$$A_1 = \frac{2\, q_\infty\, (1 - M_\infty^2 \sin^2 \psi) \cos \psi}{(K+1)\, M_\infty^2}$$

$$B_1 = -\ ln \left(\frac{\sin \psi}{1 - \cos \psi} \right) + \frac{2 + M_\infty^2 (K + \cos 2\psi)}{(1 - M_\infty^2 \sin^2 \psi) \cos \psi}$$

Making the velocities non-dimensional with respect to velocity (q_{max}), we have :

$$\bar{u} = \lambda \left[B_1 + ln \left(\frac{\sin \theta}{1 - \cos \theta} \right) - \sec \theta \right] \cos \theta \tag{10}$$

$$\bar{v} = -\lambda \left[B_1 + ln \left(\frac{\sin \theta}{1 - \cos \theta} \right) + \frac{\cos \theta}{\sin^2 \theta} \right] \sin \theta \tag{11}$$

where

$$\lambda = 2\, \bar{q}_\infty\, \frac{(1 - M_\infty^2 \sin^2 \psi) \cos \psi}{(K+1)\, M_\infty^2}$$

$$\bar{q}_\infty = \left[\frac{(K-1)\, M_\infty^2}{2 + (K-1)\, M_\infty^2} \right]^{\frac{1}{2}}$$

The resultant velocity ($\bar{q}$), can be expressed as:

$$\bar{q}^2 = \lambda^2 \left[\left\{ B_1 + ln \left(\frac{\sin \theta}{1 - \cos \theta} \right) \right\}^2 + \operatorname{cosec}^2 \theta \right] \tag{12}$$

The resultant velocity at the shock, $\bar{q}_2$, can be obtained by putting $\theta = \psi$ in (12). The relation between free-stream Mach number, M_∞, and Mach number just behind the shock, M_2, is given by :

$$M_2^2 \sin^2 \beta = \frac{1 + \frac{K-1}{2} M_\infty^2 \sin^2 \psi}{K\, M_\infty^2 \sin^2 \psi - \frac{K-1}{2}} \tag{13}$$

where

$$\sin^2 \beta = 1 - \left(\frac{\bar{q}}{\bar{q}_2} \right)^2 \cos^2 \psi$$

The relation between Mach number behind the shock (M), and the Mach number just behind the shock (M_2), is expressed as :

$$M^2 = \frac{M_2^2\, \bar{q}^2}{\bar{q}_2^2 + \frac{K-1}{2} M_2^2 (\bar{q}_2^2 - \bar{q}^2)} \tag{14}$$

where M_2 is obtained from (13).

The static pressure at any point downstream of the shock is related to the static pressure just behind the shock by :

$$(p/p_2)^{(K-1)/K} = 1 + \frac{K-1}{2} M_2^2 \left(1 - \frac{\bar{q}^2}{\bar{q}_2^2} \right) \tag{15}$$

The pressure just before and after the shock is given by well known Rankine-Hugoniot relation. The velocity components in Cartesian system are expressed as :

$$\left.\begin{aligned} \bar{u}_x &= \bar{u} \cos \theta - \bar{v} \sin \theta \\ \bar{u}_y &= \bar{u} \sin \theta + \bar{v} \cos \theta \end{aligned}\right\} \tag{16}$$

The flow defection angle ϕ is given by:

$$\tan \phi = \bar{u}_y / \bar{u}_x \tag{17}$$

In the neighbourhood of the singular line, the value of M changes very rapidly. Hence, the validity of the solution is confined in the region between the shock wave and the singular line. As a sample calculation, when $M_\infty = 6 \cdot 11$, $\alpha = 30°$, $K = 1 \cdot 4$, $\psi = 145°$, the singular line occurs when $\theta = 96 \cdot 2°$, while according to Mölder[4], it occurs at $\theta = 95 \cdot 3°$.

APPLICATIONS AND CONCLUSIONS

In external flow, the exact solutions about a sharp corner, a wedge and a cone have been used to calculate flows about bodies of complex shapes. The present solution is an internal closed form solution and can be used to check the applicability of the internal method of characteristics, tangent cone, tangent wedge, and shock expansions methods to internal flows. The present method (known as Internal Conical Flow) provides means to calculate axisymmetric duct or inlet leading edge shape for a given leading edge shock. This method shows that the solution is not valid beyond the singular lines.

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