# GEOMETRICAL RAY TRACING OF ION-TRAJECTORIES FOR AN ASTIGMATIC SECTOR FOCUSED BETA-RAY SPECTROGRAPH

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This paper describes the path of charged particles in inhomogeneous fields, having information of magnetic field at every point, and adapts the method for a high transmission double focusing spectrometer intended to enable focusing and selection of positrons for the measurement of Bhabha's exchange parameter using positron-electron scattering.

In beta-ray spectroscopy the demands for large solid angles are usually more pronounced, particularly in coincidence spectroscopy, using radio-active sources. To achieve a good luminosity alongwith a good resolution one is restricted to accept more incident flux in the median plane, rather than that in the vertical plane. Usually it is difficult to match with the theoretical field profile for a better resolution, when one uses larger polewidth, to accomplish greater transmission, alongwith a wide-gap. In this paper, it is intended to give a geometrical ray tracing method, which can be used safely, for compatible requirements of polewidth and polegap, for sector fields of double focusing type and orange spectrographs of the type given by Nielsen & Kofoed-Hansen[1]. The particular virtue of this tracing system is that it can analyse several such gaps which accept a large angle in the vertical plane, by adding them together to increase transmission in the same proportion at a given resolution, using any suitable rotationally symmetric field[2,3]. Many of the imaging relations, eg, as derived by Sturrock[4], till now have been dealing mostly with the properties of image forming by paraxial ray assemblies, where the field is assumed to be governed by the well-known relation $B_z = B_0\,(r/R)^{-n}$ [where $B_z$ is the vertical component of magnetic field at bending radius $r$, and $B_0$ the magnetic field at the central ray or reference orbit of radius $R$, whereas $n$ is the negative exponent of radial dependence of magnetic field $(-r/B)\,(dB/dr)$] for point sources only near the central ray or reference orbit. The proposed geometrical ray tracing solution has been found useful and free of such limitations for an accurate determination of imaging relations.

It will be recalled from generalized Barber's rule of magneto-optics that the bending edge, source and the image must lie on the same straight line. In other words, determination of object and image distances to a sector field (Fig 1) can be attributed to analytic properties of circles and straight lines, and a geometrical image formation technique given by Siegbahn[5] can fairly accurately describe the state of affairs except when the space-charge is predominant. For an easy short-cut Lavatelli et al.[6] and Fowler et al.[7] have been, as such, satisfied with two dimensional graphical analysis, and in a typical example,

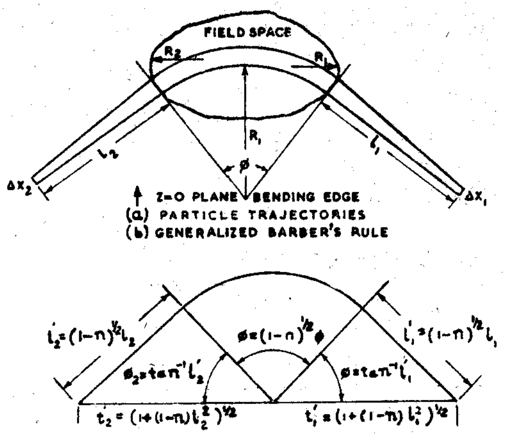


Fig 1—Radial focusing in a sector focused magnetic spectrometer.

an ion-beam in the shape of a narrow ribbon has been chosen, by Robinson[8], to solve the complexity, so that its dimension parallel to the magnetic field (here called $h$) can be assumed to be large in comparison with the maximum thickness of the beam (Fig 2) while traversing an 180° sector spectrometer. Using Newtonian mechanics (emu), the force on the particle connects the image formation when expressed as,

$$\frac{mv^2}{r} = Bev \pm 2\,(\pi\, Iec^2/vh)\,(\beta/\alpha)$$

$m$ being the mass in gm and $v$ the velocity in cm/sec of the particle moving in a cyclotron radius $r$, inside a magnetic field $B$, perpendicular to its motion, within the ion-ribbon carrying a current $I$. The $\pm$ sign refers to particles which are respectively closer and farther from the mean curvature, assuming $\alpha$ as the half angle of divergence of the beam, and $\beta$ the location of the particle in the laminum by measurement of its angular position from the centre of the beam. We can find geometrically, for a constant bending spectrometer ($R$-constant), that (Fig 2) in situations of acceptance angle $2\,\alpha \leqslant 2\,\sin^{-1}(1/6)$, (or for the ion-beams of angular spread —0·09 sterad), ions resolve themselves into a single beam (with only first order aberration having major contribution) for bending radiu of 30 cm. The centre of curvature $C'$ and $C_1$ of the rays above and below the centr orbit are found to lie very nearly in the same vertical line passing through $C$. This rou assumption although sufficient, perhaps would break down for very large (radius of cu vature) high-transmission spectrometers, where it has been found that second order abe rations, which are functions of radial and axial maximum aperture widths, contribu much. In such a special case, the method would not point out any real improvement i resolution, because it uses only first order properties of focusing. Since general equations of motion are derived in many books on electron optics, the procedure outlined will merely be for normal entry and exit cases in order that theoretical aspects of the problem may be established only. Further, effects of source width and the resultant image shapes, are not presented here on grounds of simplicity.

## UNDERLYING THEORY

Every arbitrary spatial trajectory or positron ray emitted from the source is assumed to be circle inside the field space, created by the section of the plane containing the trajectory, at an arbitrary angle, with the sphere whose centre lies on the line passing through the bending edge and perpendicular to Barber's axis (ie, the line joining the source, bending edge, and the image). We learn, therefore, that high energy particles would

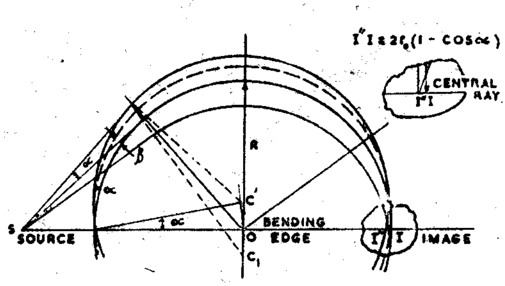


Fig 2—Two dimensional ray tracing.

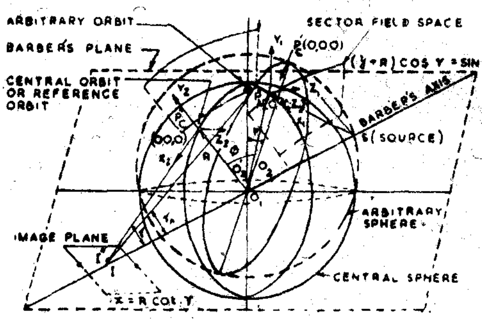


Fig 3—Three dimension analytic diagram used for ray tracing.

have to travel a circular trajectory within a sphere of greater radius and centre shifted upwards from the Barber's axis $ISO_1$ in Fig 3, and the low energy particles would move within spheres of smaller radius and having centre shifted downwards in the same vertical line. For an astigmatic sector focused magnet all these spheres would cross-over each other on a common circle, which would contain the Barber's axis as diameter, in the same plane, which could be named as Barber's plane. All these trajectories can be safely (for first order focusing) imagined as sign curves of various amplitudes relative to the "stretched-out", central ray, or reference orbit (optical axis of the system). To discuss the application of these postulates we would first give the details of the coordinate system. A simple mechanical arrangement is chosen (Fig 4), where, the end faces of the bending magnet are perpendicular to the optic axis and the beam passes symmetrically through it. Selecting the central ray as a ray that has a given position and direction at the input at a given momentum, we define the entrance coordinate system as $(X_1, Y_1, Z_1)$ with origin on the central ray, $X$-direction along the ray, $Y$-direction normal to it and in the median plane (about which the complete magnet system is mirror symmetric), and $Z$-direction perpendicular to the median plane. Similarly, in the exit space, we place a right handed coordinate system $(X_2, Y_2, Z_2)$ where again the $X$-direction coincides with the central ray after its deflection through the magnet system. Let us now determine the position and direction for an arbitrary ray in the exit space for which coordinates are $(X_1, Y_1, Z_1)$, the momentum spread

$$\delta = \Delta p/p, \text{ and slopes } y_1' = \frac{dy_1}{dx} \text{ and } z_1' = \frac{dz_1}{dx}.$$

In other words we seek the functions at $(x_2 = 0)$

$$\left.\begin{aligned} y_2 &= f_1(y_1, y_1', z_1, z_1', \delta) \quad &\text{(a)} \\ y_2' &= f_2(y_1, y_1', z_1, z_1', \delta) \quad &\text{(b)} \\ z_2 &= f_3(y_1, y_1', z_1, z_1', \delta) \quad &\text{(c)} \\ z_2' &= f_4(y_1, y_1', z_1, z_1', \delta) \quad &\text{(d)} \end{aligned}\right\} \quad (1)$$

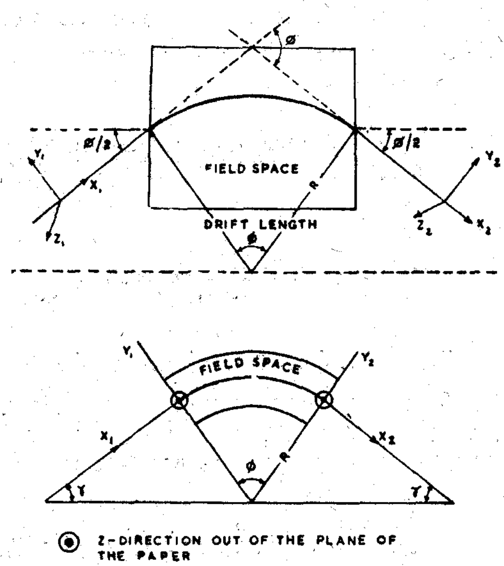


Fig 4—A general representation of beam passing symmetrically through a bending magnet with entrance coordinate $(X_1, Y_1, Z_1)$ and exit coordinates $(X_2, Y_2, Z_2)$.

where we assume that the input parameters $y_1$, $y_1'$ and $z_1$ and $z_1'$ are specified at the point where the ray passes the plane $x_1 = 0$. The parameter $\delta = \triangle p/p$ is a constant for the motion in a magnet system with stationary field. One can assume without much loss of generality (H.A. Enge-private communication) that deviations from the central ray are generally small so that $y_1'$, $z_1'$ and $\delta$ are much smaller than unity and that $y_1$ & $z_1$ both are small compared to $R$, the radius of bending of the central ray. With above assumptions it is appropriate to expand equation (1) in Taylor series. We write the series in the following form for equation 1(a).

$$y_2 = f_1(0,0,0,0,0) + \left\{ \frac{\partial f_1}{\partial y_1} y_1 + \frac{\partial f_1}{\partial y_1'} y_1' + \left( \frac{\partial f_1}{\partial z_1} z_1 + \frac{\partial f_1}{\partial z_1'} z_1' \right) + \frac{\partial f_1}{\partial \delta} \delta \right\} + \frac{1}{2!}$$

$$\left\{ \frac{\partial^2 f_1}{\partial^2 y_1} y_1^2 + \frac{\partial^2 f_1}{\partial y_1'^2} y_1'^2 + \frac{\partial^2 f_1}{\partial z_1^2} z_1^2 + \frac{\partial^2 f_1}{\partial z_1'^2} z_1'^2 + \frac{\partial^2 f}{\partial \delta^2} \delta^2 + \frac{2\partial^2 f_1}{\partial y_1 \partial y_1'} y_1 y_1' + \right.$$

$$\left( \frac{2\partial^2 f_1}{\partial y_1 \partial z_1} y_1 z_1 + \frac{2\partial^2 f_1}{\partial y_1 \partial z_1'} y_1 z_1' \right) + \frac{2\partial^2 f_1}{\partial y_1 \partial \delta} y_1 \delta + \left( \frac{2\partial^2 f_1}{\partial y_1' \partial z_1} y_1' z_1 + \frac{2\partial^2 f_1}{\partial y_1' \partial z_1'} y_1' z_1' \right)$$

$$\left. + \frac{2\partial^2 f_1}{\partial y_1' \partial \delta} y_1' \delta + \frac{2\partial^2 f_1}{\partial z_1 \partial z_1'} z_1 z_1' + \left( \frac{2\partial^2 f_1}{\partial z_1 \partial \delta} z_1 \delta + \frac{2\partial^2 f_1}{\partial z_1' \partial \delta} z_1' \delta \right) \right\} + \ldots \;\&$$

higher-order terms are neglected.

The partial derivatives, given in the parenthesis, are equal to zero because of the symmetry about the median plane which the magnet is assumed to possess. Simplifying the above equation, one gets,

$$y_2 = \left[ \frac{\partial y_2}{\partial y_1} y_1 + \frac{\partial y_2}{\partial y_1'} y_1' + \frac{\partial y_2}{\partial \delta} \delta + \tfrac{1}{2} \left\{ \frac{\partial^2 y_2}{\partial y_1^2} y_1^2 + \frac{\partial^2 y_2}{\partial y_1'^2} y'^2_1 + \frac{\partial^2 y_2}{\partial z_1^2} z_1^2 + \frac{\partial^2 y_2}{\partial z_1'^2} z_1'^2 \right. \right.$$

$$\left. \left. + \frac{\partial^2 y_2}{\partial \delta^2} \delta^2 + \frac{2\partial^2 y_2}{\partial y_1 \partial y_1'} y_1 y_1' + \frac{2\partial^2 y_2}{\partial y_1' \partial \delta} y_1' \delta + \frac{2\partial^2 y_2}{\partial z_1 \partial z_1'} z_1 z_1' \right\} \right]$$

The expansions have been carried only to second-order. The partial derivatives that appear in Taylor expansion can be written in abbreviated form using Enge's notation[9] and can be called as "focusing coefficients". These parameters are further made non-dimensional by dividing $y_2$ and $z_2$ with the reference radius $R$.

$$y_2/R = (y/y) \frac{y_1}{R} + (y/y') y_1' + (y/\delta)\delta + (y/y^2)(y_1/R)^2 + (y/yy') \frac{y_1}{R} y_1' + (y/y\delta) \frac{y_1}{R} \delta +$$
$$+ (y/y'^2) y_1'^2 + (y/y'\delta) y_1'\delta + (y/\delta^2)\delta^2 + (y/z^2)(z_1/R)^2 + (y/zz')(z_1/R) z_1' +$$
$$+ (y/z'^2) z_1'^2 \qquad (2)$$

For example, the meaning of some coefficients in equation (2) in terms of notation used in regular series is $(y/y) = \partial y_2/\partial y_1$, $(y/y') = \dfrac{1}{R}\dfrac{\partial y_2}{\partial y_1'}$ and $(y/y'^2) = \dfrac{R^2}{2\,!}\dfrac{\partial^2 y_2}{\partial y_1'^2}$.

Some coefficients eg $(y/yz) = 0$. Similarly expressing other functions,

$$\begin{aligned} y_2' =\ & (y'/y)\ (y_1/R) + (y'/y')\ y_1' + (y'/\delta)\ \delta + (y'/y^2)\ (y_1/R)^2 + (y'/yy')\ (y_1/R)\, y_1' + \\ & (y'/y\delta)\ (y_1/R)\,\delta + (y'/y'^2)\, y_1'^2 + (y'/y'\delta)\ y_1'\delta + (y'/\delta^2)\,\delta^2 + (y'/z^2)\,(z_1/R)^2 + \\ & (y'/zz')\,(z_1/R)\,(z_1') + (y'/z'^2)\, z_1'^2 \end{aligned} \tag{3}$$

$$\begin{aligned} z_2/R = & (z/z)\,(z_1/R) + (z/z')\, z_1' + (z/yz)\,(y_1/R)\,(z_1/R) + (z/y'z)\, y_1'\,(z_1/R) + (z/yz')\,(y_1/R) \\ & z_1' + (z/y'z')\, y_1' z_1' + (z/\delta z_1')\,\delta z_1' \end{aligned} \tag{4}$$

$$\begin{aligned} z_2' =\ & (z'/z)\,(z_1/R) + (z'/z')\ z_1' + (z'/yz)\ (y_1/R)\ (z_1/R) + (z'/y'z')\ y_1'\ (z_1/R) + (z'/\delta z) \\ & \delta\ (z_1/R) + (z'/yz')\, z_1'\,(y_1/R) + (z'/y'z')\, y_1' z_1' + (z'/\delta z')\ \delta z_1' \end{aligned} \tag{5}$$

as equations (3), (4) and (5), we get the exit parameters and a knowledge about focusing and dispersing properties of a magnet system whose field can be described by the expression:

$$\frac{B}{B_0} = 1 - n\rho + \beta\rho^2 + \gamma\rho^3, \text{ where } \rho = (r-R)/R \tag{6}$$

The field coefficients $n$, $\beta$ and $\gamma$ are functions of geometry and permeability of the spectrometer's pole-pieces and to a lesser extent to that of the yoke. With the exit parameters defined, we would proceed to determine the ray behaviour in space using assumptions postulated before alongwith generalized Barber's rule of magneto-optics. Let us choose an arbitrary ray forming a part of a circular trajectory contained in the dotted sphere of radius $r$, within the magnet system. Equation of such a sphere with its centre shifted upwards by a distance $\xi$ from the centre of the sphere containing the central ray $(O_1O_2' = \xi)$ is,

$$(x+\xi \sin\gamma)^2 + (y+R-\xi\cos\gamma)^2 + z^2 = r^2 \tag{7}$$

Consider now the plane through the Barber's axis and let it be assumed as Barber's plane, given by the expression,

$$(y+R)\cos\gamma = x\sin\gamma,\ z = 0.$$

Therefore, the plane making an angle $\psi$ with $X_2\ Y_2$ plane can be expressed as

$$y+R - x\tan\gamma + lz = 0, \quad \text{where } l = \cos\psi.$$

The intersection point $P_A''$ $(0, y_2, z_2)$ of the arbitrary circular orbit with plane $x_2 = 0$, can be put as

$$z_2 = -\frac{l\,\xi\cos\gamma + (-\xi^2 l^2 \sin^2\gamma - \xi^2 + r^2 + r^2 l^2)^{\frac{1}{2}}}{1+l^2}$$

Assuming abbreviations

$$q = \xi \, l \cos \gamma \text{ and } \lambda = \{-\xi^2 (1 + \sin^2 \gamma) + r^2 (1+l^2)\}^{\frac{1}{2}}$$

and on simplifying, we get

$$z_2 = -\frac{q + \sqrt{\lambda}}{1+l^2}$$

From the geometry of the Fig. 3, we get, similarly,

$$y_2 = -R - \{z_1 l = -R - l\,(-q+\sqrt{\lambda})/(1+l^2)\} = -R + \frac{ql - l\sqrt{\lambda}}{1+l^2}$$

The exit coordinates sought before are hence equivalent to the coordinates of the point

$$P''_A\left(0, -R + \frac{ql - l\sqrt{\lambda}}{1+l^2}, -\frac{q + \sqrt{\lambda}}{1+l^2}\right).$$

Now to get the exact situation of the image $I''$, one faces an obvious necessity of calculating direction cosines of the tangent at $P_A''$, being given as,

$$\frac{dx}{ds} \propto l\,(y + R - \xi \cos \gamma) - z,$$

which can be simplified as $-\sqrt{\lambda}$

Similarly

$$\frac{dy}{ds} \propto -z \tan \gamma - l\,(x + \xi \sin \gamma)$$

can be simplified by proper substitution as

$$\frac{dy}{ds} \propto \tan \gamma \,[(-\sqrt{\lambda} - ql^2)\,/\,(1 + l^2)]$$

and

$$\frac{dz}{ds} \propto \tan \gamma \,[(ql - l\sqrt{\lambda})\,/\,(1 + l^2)]$$

would give

$$\frac{dx}{ds} : \frac{dy}{ds} : \frac{dz}{ds} = \sqrt{\lambda}\,(1+l^2) : \tan \gamma\,(ql^2 + \sqrt{\lambda}) : \tan \gamma\,(l\sqrt{\lambda} - ql)$$

Tangent at $P''_A\,(0, y_2, z_2)$ is given by

$$\frac{x - 0}{\sqrt{\lambda}\,(1+l^2)} = \frac{y - y_2}{\tan \gamma\,(ql^2 + \sqrt{\lambda})} = \frac{z - z_2}{\tan \gamma\,(l\sqrt{\lambda} - ql)} = \text{constant} = C \qquad (8)$$

Now in order to evaluate the focusing properties of the magnet system we find the intersection of the tangent with the plane parallel to $Y_2Z_2$ plane, given by $x = R \cot \gamma$. The point $I$, where the central ray meets the image plane and the Barber's axis has coordinates $(R \cot \gamma, 0, 0)$. The coordinates of $I''$ is given as,

$$x = R \cot \gamma$$

$$y = y_2 + C \tan \gamma \, (ql^2 + \sqrt{\lambda})$$

From the equation (8) we get by proper substitution,

$$C\sqrt{\lambda}\,(1 + l^2) \qquad = R \cot \gamma \text{ or } C = R \cot \gamma / \sqrt{\lambda}\,(1 + l^2)$$

$$y = -R + \frac{ql - l\sqrt{\lambda}}{1 + l^2} + \frac{R \cot \gamma}{\sqrt{\lambda}\,(1+l^2)} \tan \gamma \, (ql^2 + \sqrt{\lambda})$$

$$= l\,(q - \sqrt{\lambda})\,(Rl + \sqrt{\lambda}) \,/\, \sqrt{\lambda}\,(1+l^2)$$

and
$$z = z_2 + C \tan \gamma \, (l\sqrt{\lambda} - ql)$$

$$= -\frac{q + \sqrt{\lambda}}{1 + l^2} + \frac{R \cot \gamma}{\sqrt{\lambda}\,(1 + l^2)} \tan \gamma \, (l\sqrt{\lambda} - ql)$$

$$= -(q - \sqrt{\lambda})\,(\sqrt{\lambda} + Rl) \,/\, \sqrt{\lambda}\,(1 + l^2)$$

$$II''^2 = 0^2 + \frac{l^2\,(q - \sqrt{\lambda})^2\,(Rl + \sqrt{\lambda})^2}{\lambda\,(1 + l^2)^2} + \frac{(q - \sqrt{\lambda})^2\,(Rl + \sqrt{\lambda})^2}{\lambda\,(1 + l^2)^2}$$

Wherefrom, $$II'' = \frac{(q - \sqrt{\lambda})\,(\sqrt{\lambda} + Rl)}{\sqrt{\lambda}\,(1 + l^2)}$$

gives the size of the focused image for any arbitrary ray, as bent and selected by a sector focused field. The momentum spread becomes

$$\delta = \triangle p/p = II''/R\,D_p$$

where $D_p$ is the momentum dispersion accepted by the magnet system in order to converge the arbitrary ray. The resolving power (RP), can similarly, be given as

$$\text{RP} = p/\triangle p = R\,D_p/II''$$

If the magnet has a low momentum spread, the value $\triangle p$ is momentum half-width in that case. If the source is large, then, a rough estimate shows $II'' = M_H\,S_1$, where $S_1$ is the source width and $M_H$ the magnification in the median plane, or horizontal plane. Accordingly one would expect,

$$\triangle p = p_0\,M_H\,S_1/D_p\,R$$

where $D_p/M_H$, $S_1$, and $R$ being known parameters, one can evaluate $\triangle p$ easily because the central ray momentum can be known from a calibration of the spectrometer for its linearity and the knowledge of magnetic rigidity curve.

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