EXPANSION FORMULAE FOR KAMPÉ DE FÉRIET AND RADIAL WAVE FUNCTIONS AND HEAT CONDUCTION

F. SINGH

Government Engineering College, Rewa (M. P.)

(*Received* 29 *December*, 1970)

Author evaluates some integrals involving Kampé de Fériet function and uses one of them in obtaining a solution of a problem of heat conduction. Three expansion formulae for Kampé de Fériet function have also been obtained and a few interesting cases along with the application of one of them, in the radial wave function for the hydrogen like atoms, have also been discussed.

The present paper is inspired by the frequent requirement of various properties of special functions which play a vital role in the study of potential and other allied problems in quantum mechanics. Appell's functions and the functions related to them have many applications in mathematical physics[1-3] and author evaluates here some integrals involving Kampé de Fériet function and one of them has been employed to obtain a solution of a problem in heat conduction given by Bhonsle[4]. Three expansion formulae for Kampé de Fériet function have also been obtained. A few particular cases of interest from the point of view of their applicability in quantum mechanics, have also been discussed.

We make use of familiar abbreviation.

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a\,(a+1)\;\ldots\ldots\;(a+m-1);$$

and in what follows for the sake of brevity and elegance we express the Kampé de Fériet function in the notations of Burchnall & Chaundy[5].

$$F\left[\begin{matrix}(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix}\;x,\,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a))_{m+n}\,((a'))_{m+n}\,((c))_m\,((c'))_n\,x^m y^n}{((b))_{m+n}\,((b'))_{m+n}\,((d))_m\,((d'))_n\,(m)!\,(n)!},$$

where $$A + A' + C \leqslant B + B' + D, \quad A + A' + C' \leqslant B + B' + D'. \tag{1}$$

Further (a) is taken to denote the sequence of A parameters $a_1, \ldots\ldots\ldots\ldots, a_A$, that is, unless stated otherwise there are A of a parameters, A' of a' parameters and so on.

Thus $((a))_m$ is to be interpreted as $\prod_{j=1}^{A} (a_j)_m$, with similar interpretations for $((a))_m$ etc.

Following formulae are required in the proof of the integrals :

$$\int_{-\infty}^{\infty} z^{2\rho}\, e^{-z^2}\, H_{2\nu}(z)\, dz = \frac{\sqrt{\pi}\,.\,2^{2(\nu-\rho)}\,\Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)}, \quad \rho = 0,\,1,\,2,\,\ldots\ldots \tag{2}$$

The integral follows by multiplying both sides Lebdev[6] equation (4·16·1) by $e^{-z^2} H_{2\nu}(z)$, integrating with respect to z from $-\infty$ to ∞ and using the orthogonal property of Hermite polynomials[7].

$$\int_0^\infty z^\rho e^{-z} L_n^\alpha(z)\, dz = \frac{\Gamma(\alpha-\rho+n)\ \Gamma(\rho+1)}{(n)!\ \Gamma(\alpha-\rho)}, \quad Re\,(\rho+1) > 0 \tag{3}$$

and

$$\int_{-1}^{1} (1-z^2)^{\rho-1} P_\nu^\mu(z)\, dz = \frac{\pi\, 2^\mu\ \Gamma\left(\rho+\frac{\mu}{2}\right)\ \Gamma\left(\rho-\frac{\mu}{2}\right)}{\Gamma\left(1+\rho+\frac{\nu}{2}\right)\Gamma\left(\rho-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}+1\right)\Gamma\left(-\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2}\right)}, \tag{4}$$

$$2\, Re\,(\rho) > |\, Re\,(\mu)\,|.$$

Equations (3) and (4) are known results[8].

DERIVATION OF THE INTEGRALS

Here we obtain the following integrals to be used later in obtaining the solution of the heat conduction problem and expansion formulae.

$$\int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z)\ F\begin{bmatrix}(a), (a') : (c) ; (c') ; \\ (b), (b') : (d) ; (d') ;\end{bmatrix} xz^{2h},\ yz^{2h} \Big]\, dz$$

$$= 2^{2\nu} M_\nu^h\ F\begin{bmatrix}(a), (a'), \triangle(2h, 1+2\rho) : (c) ; (c') ; \\ (b), (b'), \triangle(h, 1+\rho-\nu) : (d) ; (d') ;\end{bmatrix} xh^h,\ yh^h \Big], \tag{5}$$

where h is a positive integer,

$$A + A' + C \leqslant B + B' + D;$$
$$A + A' + C' \leqslant B + B' + D';$$
$$\rho = 0, 1, 2, \ldots ; \ \triangle(m, a)$$

represents

$$\frac{a}{m}, \ \frac{a+1}{m}, \ \ldots\ldots, \ \frac{a+m-1}{m}$$

and
$$M_\nu^h = (2\pi)^{\frac{1}{2}(1-h)}\ h^\rho\ \frac{\prod_{j=0}^{2h-1} \Gamma\left(\frac{1+2\rho+j}{2h}\right)}{\prod_{j=0}^{h-1} \Gamma\left(\frac{1+\rho-\nu+j}{h}\right)}$$

$$\int_0^\infty z^\rho e^{-z} L_\mu^\alpha(z)\ F\begin{bmatrix}(a), (a') : (c) ; (c') ; \\ (b), (b') : (d) ; (d') ;\end{bmatrix} xz^h,\ yz^h \Big]\, dz$$

$$= \frac{A_u^\rho}{(u)!}\ F\begin{bmatrix}(a), (a'), \triangle(h, 1+\rho), \triangle(h, \alpha-\rho) : (c) ; (c') ; \\ (b), (b'), \triangle(h, \alpha+u-\rho) : (d) ; (d') ;\end{bmatrix} xh^h,\ yh^h \Big], \tag{6}$$

provided h is a positive integer,

$$A + A' + C \leqslant B + B' + D,$$
$$A + A' + C' \leqslant B + B' + D',$$
$$Re\,(1 + \rho) > 0$$

and

$$A_u^{\rho} = \frac{(2\pi)^{\frac{1}{2}(1-h)}\, h^{\rho+u+\frac{1}{2}} \prod_{j=0}^{h-1} \Gamma\left(\frac{1+\rho+j}{h}\right) \prod_{j=0}^{h-1} \Gamma\left(\frac{1+\rho-\alpha-u+j}{h}\right)}{\prod_{j=0}^{h-1} \Gamma\left(\frac{1+\rho-\alpha+j}{h}\right)};$$

and

$$\int_{-1}^{1} (1-z^2)^{\rho-1} P_\nu^\mu(z)\, F\left[\begin{matrix}(a), & (a'): (c); (c'); \\ (b), & (b'): (d); (d');\end{matrix}\; x(1-z^2)^h,\; y(1-z^2)^h\right] dz$$

$$= B_\nu\, F\left[\begin{matrix}(a), (a'), \triangle\left(h, \rho + \frac{\mu}{2}\right), \triangle\left(h, \rho - \frac{\mu}{2}\right) : (c); (c'); \\ (b), (b'), \triangle\left(h, 1 + \rho + \frac{\nu}{2}\right), \triangle\left(h, \rho - \frac{\nu}{2}\right) : (d); (d');\end{matrix}\; x, y\right] \tag{7}$$

provided that h is a positive integer,

$$A + A' + C \leqslant B + B' + D,$$
$$A + A' + C' \leqslant B + B' + D',$$
$$2\,Re\,(\rho) > |\,Re\,(\mu)\,|$$

and

$$B_\nu = \frac{\pi\, 2^{\mu} \prod_{j=0}^{h-1} \Gamma\left(\frac{\rho + \frac{\mu}{2} + j}{h}\right) \prod_{j=0}^{h-1} \Gamma\left(\frac{\rho - 1 - \frac{\mu}{2} + j}{h}\right)}{h\, \Gamma\left(\frac{\nu}{2} - \frac{\mu}{2} + 1\right) \Gamma\left(-\frac{\nu}{2} - \frac{\mu}{2} + \frac{1}{2}\right) \prod_{j=0}^{h-1} \Gamma\left(\frac{\rho + \frac{\nu}{2} + 1 + j}{h}\right) \prod_{j=0}^{h-1} \Gamma\left(\frac{\rho - \frac{\nu}{2} + j}{h}\right)}.$$

To establish the integrals (5) to (7), express the Kampé de Fériet function on the L. H. S. in the series form and then change the order of integration and summations which is justified under the conditions given, evaluate the inner integrals with the help of equations (2) to (4) respectively and simplify with the help of multiplication formula for gamma function[7].

HEAT CONDUCTION AND KAMPE' DE FE'RIET FUNCTION

Hermite polynomials have been utilized by Kampé de Fériet[9] in solving a heat conduction equation. He has obtained rour theorems which are of the nature of existence theorems. Recently Bhonsle[4] has employed Hermite polynomials in solving the partial differential equation

$$\frac{\partial \phi}{\partial t} = K \frac{\partial^2 \phi}{\partial z^2} - K \phi z^2, \tag{8}$$

where $\phi(z, t)$ tends to zero for large values of t and when $|z| \to \infty$, this equation is related to the problem of heat conduction[10]

$$\frac{\partial \phi}{\partial t} = K \frac{\partial^2 \phi}{\partial z^2} - h_1(\phi - \phi_0), \tag{9}$$

provided that

$$\phi_0 = 0 \text{ and } h_1 = Kz^2.$$

The solution of (8) given by Bhonsle[4] is

$$\phi(z, t) = \sum_{r=0}^{\infty} Q_r e^{-(1+2r)Kt - \frac{z^2}{2}} H_r(z) \cdot \tag{10}$$

We shall consider the problem of determining a function $\phi(z, t)$; if $t = 0$, then

$$\phi(z, 0) = z^{2\rho} e^{-z^2} F \left[\begin{matrix} (a), (a') : (c) ; (c') ; \\ (b), (b') : (d) ; (d') ; \end{matrix} \; xz^{2h}, yz^{2h} \right]. \tag{11}$$

If $t = 0$, then by virtue of (10) and (11), we have

$$z^{2\rho} e^{-} F \left[\begin{matrix} (a), (a') : (c) ; (c') ; \\ (b), (b') : (d) ; (d') ; \end{matrix} \; xz^{2h}, yz^{2h} \right] = \sum_{r=0}^{\infty} Q_r e^{-\frac{z^2}{2}} H_r(z). \tag{12}$$

Multiplying both sides of (12) by $H_\mu(z)$ and integrating with respect to z between $-\infty$ to ∞, we have

$$\int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_\mu(z) F \left[\begin{matrix} (a), (a') : (c) ; (c') ; \\ (b), (b') : (d) ; (d') ; \end{matrix} \; xz^{2h}, yz^{2h} \right] dz$$

$$= \sum_{r=0}^{\infty} Q_r \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H_\mu(z) H_r(z) \, dz. \tag{13}$$

Using (5) and the orthogonal property of Hermite polynomials[8], we obtain

$$Q_\mu = \frac{2^\mu M_\mu^h}{(\mu)! \, (2\pi)^{\frac{1}{2}}} F \left[\begin{matrix} (a), (a'), \triangle(2h, 1 + 2\rho) ; (c); (c'); \\ (b), (b'), \triangle(h, 1 + \rho - \frac{\mu}{2}) : (d); (d') ; \end{matrix} \; xh^h, yh^h \right]. \tag{14}$$

Now with the help of (14), the solution (10) reduces to the form

$$\phi(z,t) = \sum_{r=0}^{\infty} \frac{2^{r-\frac{1}{2}} M_r^h \, e^{-(1+2r)Kt - \frac{z^2}{2}}}{(r)! \sqrt{\pi}} F \left[\begin{matrix} (a), (a'), \triangle(2h, 1 + 2\rho) : (c); (c'); \\ (b), (b'), \triangle(h, 1 + \rho - r/_2) : (d); (d') ; \end{matrix} \; xh^h, yh^h \right].$$

$$\cdot H_r(z). \tag{15}$$

The conditions of validity are the same as specified in (5).

EXPANSION FORMULAE

The expansions to be established here are

$$z^{2\rho}e^{-z^2/2}\, F\left[\begin{matrix}(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix}\, xz^{2h}, yz^{2h}\right]$$

$$=\sum_{r=0}^{\infty}\frac{2^{r-1/2}M_r^h}{(r)!\sqrt{\pi}}F\left[\begin{matrix}(a),(a'),\triangle(2h,1+2\rho):(c);(c');\\(b),(b'),\triangle(h,1+\rho-r/2):(d);(d');\end{matrix}\, xh^h, yh^h\right]H_r(z), \tag{16}$$

the conditions of validity are the same as for (5).

$$z^{\rho}\, F\left[\begin{matrix}(a),(a'):(c);(c');\\(b),(b'):(d);(d')\end{matrix}\; xz^h,\; yz^h\right]$$

$$=\sum_{r=0}^{\infty}\frac{A_r^{\rho+\alpha}}{\Gamma(1+\alpha+r)}F\left[\begin{matrix}(a),(a'),\triangle(h,1+\alpha+\rho),\triangle(h,-\rho):(c);(c');\\(b),(b'),\triangle(h,r-\rho):(d);(d');\end{matrix}\, xh^h, yh^h\right]L_r^{\alpha}(z), \tag{17}$$

which is valid under the same conditions as specified in (6) along with $\rho \geqslant 0$ and $Re(\alpha)>0$.

$$(1-z^2)^{\rho-1}\, F\left[\begin{matrix}(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix}\, x(1-z^2)^h, y(1-z^2)^h\right]$$

$$=\sum_{r=0}^{\infty}B_r\frac{(2r+1)(r-\mu)!}{2(r+\mu)!}P_r^{\mu}(z)\,F\left\{\begin{matrix}(a),(a'),\triangle(h,\rho+\mu/2),\triangle(h,\rho-\mu/2):(c);(c');\\(b),(b'),\triangle(h,1+\rho+\nu/2),\triangle(h,\rho-\nu/2):(d);(d');\end{matrix}\, x,y\right\}. \tag{18}$$

The conditions of validity for (18) are the same as for (7) along with $\rho \geqslant 1$.

Proof : The expansion (16) can be established in the light of the equation (12) and (14). To prove (17) let

$$f(z)=z^{\rho}\, F\left[\begin{matrix}(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix}\; xz^h,\; yz^h\right]=\sum_{r=0}^{\infty}Q_r\, L_r^{\alpha}(z)\,. \tag{19}$$

Equation (19) is valid since $f(z)$ is continuous and of bounded variation in the open interval $(0, \infty)$ when $\rho \geqslant 0$. Multiplying both sides of (19) by $z^{\alpha}e^{-z}L_u^{\alpha}(z)$, integrating with respect to z from 0 to ∞ and finally using the result (6) and orthogonal property of Laguerre polynomials[8], we get

$$Q_u=\frac{A_u^{\rho+\alpha}}{\Gamma(1+\alpha+u)}\,F\left[\begin{matrix}(a),(a'),\triangle(h,1+\alpha+\rho),\triangle(h,-\rho):(c);(c');\\(b),(b'),\triangle(h,u-\rho):(d);(d');\end{matrix}\; xh^h,\; yh^h\right]. \tag{20}$$

Hence the expansion (17) follows immediately from (19) and (20).

The proof of the expansion (18) is parallel to that of (17) and is obtained on using (7) and the orthogonal property of associated Legendre function[11] viz.

$$\int_{-1}^{1}P_n^{m}(x)\,P_r^{m}(x)\,dx=\begin{cases}0, & (r\neq n)\\[4pt] \dfrac{2(m+n)!}{(2n+1),(n-m)!}, & (r=n)\,.\end{cases}$$

EXPANSION OF RADIAL WAVE FUNCTION

In equation (16) if $a = b$ and $a' = b'$, the double hypergeometric function on the left breaks up into the product of two generalized hypergeometric functions, thus,

$$z^{2\rho} e^{-z^2/2} {}_C F_D \left[\begin{matrix} (c) ; \\ (d) ; \end{matrix} \; xz^{2h} \right] {}_{C'}F_{D'} \left[\begin{matrix} (c') ; \\ (d') ; \end{matrix} \; yz^{2h} \right]$$

$$= \sum_{r=0}^{\infty} \frac{2^{r-\frac{1}{2}} M_r^h}{(r)! \sqrt{\pi}} F \left[\begin{matrix} \triangle (2h, 1+2\rho) : (c) ; (c') ; \\ \triangle (h, 1+\rho-r/2) : (d) ; (d') ; \end{matrix} \; xh^h, yh^h \right] H_r(z). \qquad (21)$$

The conditions of validity for (21) are the same (with $A = B$ and $A' = B'$) as for (5).

If $y = 0$; the special case $A = A' = B = B' = 0$ of (16) yields

$$z^{2\rho} e^{-z^2/2} {}_C F_D \left[\begin{matrix} (c) ; \\ (d) ; \end{matrix} \; xz^{2h} \right]$$

$$= \sum_{r=0}^{\infty} \frac{2^{r-\frac{1}{2}} M_r^h}{(r)! \sqrt{\pi}} {}_{c+2h}F_{D+h} \left[\begin{matrix} \triangle \; (2h, 1+2\rho), (c) ; \\ \triangle \; (h, 1+\rho-r/2), (d) ; \end{matrix} \; xh^h \right] H_r(z), \qquad (22)$$

which exists under conditions given in (5) with $A = A' = B = B' = C' = D' = 0$.

We know[12] that the normalized radial part of the wave function for the Hydrogen atom is

$$R_{nl}(\gamma) = - \left[\left(\frac{2z}{na_0} \right)^3 \frac{(n-l-1)!}{2n \{(n+l)!\}^3} \right]^{\frac{1}{2}} e^{-\frac{z\gamma}{na_0}} \left(\frac{2z\gamma}{na_0} \right)^l L_{n+l}^{2l+1} \left(\frac{2z\gamma}{na_0} \right). \qquad (23)$$

Changing the associated Laguerre function in (23) into the confluent hypergeometric function[13], we obtain

$$R_{nl}(\gamma) = -\left(\sqrt{\frac{z}{n^2 a_0} \left(\frac{1}{\gamma^2} \right) \frac{(n+l)!}{(n-l-1)!}} \right) \cdot \frac{e^{\frac{z\gamma}{na_0}}}{(2l+1)!} \left(\frac{2z\gamma}{na_0} \right)^{l+1} \cdot$$

$$\cdot {}_1F_1 \left(-n+l+1 ; 2l+2 ; \frac{2z\gamma}{na_0} \right). \qquad (24)$$

Now setting $C = D = 1$, $C_1 = -n+l+1$, $d_1 = 2l+2$, $h = 1$, $\rho = 0$, replacing z^2 by t and x by $\frac{2z}{na_0}$ in (22), we obtain finally with the help of (24).

$$R_{nl}(t) = - \sqrt{\frac{z(n+l)!}{n^2 a_0 (n-l-1)!}} \cdot \frac{e^{t\left(\frac{1}{2} - \frac{z}{na_0}\right)}}{(2l+1)!} \left(\frac{2z}{na_0} \right)^{l+1} t^l .$$

$$\cdot \sum_{r=0}^{\infty} \frac{2^{r-1/2} M_r^1}{(r)! \sqrt{\pi}} \cdot {}_3F_2 \left[\begin{matrix} -n+l+1, \frac{1}{2}, 1 ; \\ 2l+2, 1-\frac{r}{2} ; \end{matrix} \; \frac{2z}{na_0} \right] H_r(\pm\sqrt{t}) ; \qquad (25)$$

where $n \geq l+1$.

Although the hydrogen like radial wave functions appear to be very complicated, they actually reduce to relatively simple forms, especially for low values of total quantum number n and azimuthal quantum number l. The expressions for $R_{nl}(t)$ computed from (25) for $n = 1$ and 2, $l = 0$ and 1 are given in Table 1.

TABLE 1

NORMALIZED RADIAL FUNCTIONS FOR HYDROGEN-LIKE ATOMS

n	l	$R_{nl}(t)$
1	0	$-\left(\frac{z}{a_0}\right)^{\frac{3}{2}} e^{t\left(\frac{1}{2}-\frac{z}{a_0}\right)} \sum_{r=0}^{\infty} \frac{2^{r+1/2} M_r^1}{(r)!\sqrt{\pi}} H_r(\pm\sqrt{t})$.
2	0	$-\left(\frac{z}{a_0}\right)^{\frac{3}{2}} e^{t\left(\frac{1}{2}-\frac{z}{2a_0}\right)} \sum_{r=0}^{\infty} \frac{2^{r-1} M_r^1}{(r)!\sqrt{\pi}} H_r(\pm\sqrt{t}) \left\{1-\frac{z}{z\, a_0(2-r)}\right\}$.
2	1	$-\left(\frac{z}{a_0}\right)^{\frac{5}{2}} e^{t\left(\frac{1}{2}-\frac{z}{2a_0}\right)} t \sum_{r=0}^{\infty} \frac{2^{r-2} M_r^1}{\sqrt{3}\,(r)!\sqrt{\pi}} H_r(\pm\sqrt{t})$.

CONCLUSIONS

It may be of interest to conclude that on reducing, the generalized hypergeometric functions on the L. H. S. of expansions (21) and (22) yield some very interesting results. This fact is established in the light of the results[7] :

$$ {}_2F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2}b,\ \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\,; \\ a,\ b,\ a+b-1\,;\end{matrix}\ 4x\right] = {}_0F_1\left[\begin{matrix}-\,; \\ a\,;\end{matrix}\ x\right] {}_0F_1\left[\begin{matrix}-\,; \\ b\,;\end{matrix}\ x\right] $$

and

$$ {}_2F_3\left[\begin{matrix}a,\ b-a\,; \\ b,\ \frac{1}{2}b,\ \frac{1}{2}b+\frac{1}{2}\,;\end{matrix}\ \tfrac{1}{4}x^2\right] = {}_1F_1\left[\begin{matrix}a\,; \\ b\,;\end{matrix}\ x\right] {}_1F_1\left[\begin{matrix}a\,; \\ b\,;\end{matrix}\ -x\right]. $$

By proper choice of parameters, ${}_0F_1$ can be reduced to Bessel function or transformed to ${}_1F_1$ by Kummer's second theorem[7]. Further ${}_1F_1$ can be reduced to Whittaker function $M_{k,m}(x)$, generalized Laguerre polynomial $L_n^{\alpha}(x)$, Hermite, polynomial $H_n(x)$, and regular and irregular Coulomb wave functions F_L and G_L, thereby providing us with such results as may be used in various problems encountered in Quantum mechanics viz., collision problem or two particles with Coulomb interaction, Harmonic oscillator and the Hydrogen atom[14]. The results similar to those obtained under "Expansion of Radial Wave Function", may also be obtained from the expansions (17) and (18). It may be further remarked that Kampé de Fériet function not only reduces to generalized hypergeometric function or the product of two generalized hypergeometric functions but it also yields Appell's functions F_1, F_2, F_3 and F_4.

Thus the results evaluated in this paper find application not only in the derivation of "Radial Wave Function" but are also useful in obtaining many new results involving the generalized hypergeometric functions and the Appell's functions.

REFERENCES

1. JAEGER, J. C., A continuation formula for Appell's function F_2, *J. Lond. Math. Soc.*, 13 (1938).
2. NAGEL, B., OLSSON, P. & WEISSGLAS, P., *Ark. Fys.*, 23 (1963), 137-143.
3. OLLSSON, P. O. M., A hypergeometric Function of Two Variables of Importance in Perturbation Theory, I, II, *Ark. Fys.*, 30 (1965), 187-191, 29, 459-465.
4. BHONSLE, B. R., *Proc. Nat. Acad. Sci. India*, 36 (1966), 356-360.
5. BURCHNALL, J. L. & CHAUNDY, T. Y., *Quart. J. Math.*, Oxford Series, 12 (1941), 112-128.
6. LEBDEV, N. N., "Special Functions and their Application", (Prentice Hall, Inc. Engle Wood, Cliffs, N. J.) 1965, p. 74.
7. RAINVILLE, E. D., "Special Functions", 1965, pp. 192-193; 24; 105-106, 126.
8. BATEMAN project: "Tables of Integral Transforms, Vol. II", (MacGraw-Hill) 1954, pp. 292-3, 316, 289.
9. KAMPE' DE FE'RIET, J., Heat Conduction and Hermite Polynomials, *Bull. Calcutta Math. Soc.*, The Golden Jubilee Commemoration Volume, 1958-9, 193-204.
10. CHURCHIL, R. V., "Operational Mathematics", (MacGraw-Hill, New York) 1958, p. 130.
11. WHITTAKER, E. T. & WATSON, G. N., "A Course of Modern Analysis", (Cambridge), 1952, p. 324.
12. PAULING, L. & WILSON, E. B., "Introduction to Quantum Mechanics". (MacGraw-Hill) 1935.
13. SNEDDON, I. N., Special Functions of Mathematical Physics and Chemistry, Interscience, 1961, p. 166.
14. MESSIAH, A. "Quantum Mechanics, Vol. 1", (North Holland Pub. Co.) 1967, p. 491, 415, 421.