DETERMINATION OF OPTIMUM NON-SLENDER BODIES OF REVOLUTION HAVING MINIMUM DRAG IN FREE-MOLECULAR FLOW

V. B. Tawakley & S. C. Jain

Defence Science Laboratory, Delhi

(*Received* 28 *December*, 1970)

The problem of determining the body of revolution having minimum drag has been considered in the free molecular flow region under given integral constraints and in combination with the calculus of variations. It has been shown that for a given surface area or volume, the optimum shape is sharp nosed if diameter is prescribed and it is flat nosed if the length is prescribed. When the volume and thickness are known the optimum shape has cusp at the origin whereas if surface area and diameter are given the optimum body shape is independent of the value of k, a parameter dependent upon the temperature ratio and the speed ratio.

There have been several studies on the problem of determining the aerodynamic shapes of bodies having minimum drag under different geometric constraints on the body in the recent past studies[1]. Most of these are in the 'Continum flow' regime, ie, in the flow which is such that the mean free path is small with respect to a characteristic dimension of the body. But in case of 'free molecular flow', the ratio of the mean free path to a characteristic dimension of the body is large compared to unity and as such the aerodynamic drag is governed by the interaction of the impinging molecules and the surface. It is generally assumed that the molecules hitting the surface are first absorbed and then remitted with Maxwellian velocity distribution and that the temperature of the remitted molecules is identical with the surface temperature. The drag so obtained has been used by some authors[2-8] to the problem of determining the body of revolution having minimum drag with the constraints that the length and the thickness of the body are specified. In these analyses the body has been assumed to be slender, ie, the value of the thickness ratio is such that $\tau << 1$. Tan[9] has studied the problem for the non-slender body of revolution for a given length and a given diameter. These results are necessarily limited in the sense that the length and the diameter of a body are not the only relevant parameters when considering optimum body profiles. It is, therefore, essential to extend the analysis when there are integral constraints on the body. Here we consider the problem of finding the non-slender body of revolution of minimum drag in free molecular flow under integral constraints, ie, when either the surface area or the volume of the body is essentially specified.

GENERAL FORMULATION OF THE PROBLEM

The general problem may be defined as follows:

Suppose it is required to extremise an integral of the form

$$J_1 \equiv f_1\ (y_i,\ y_f) + \int_{x_i}^{x_f} \phi_1\ (y, \dot{y})\ dx \tag{1}$$

with respect to the class of arcs satisfying certain prescribed boundary conditions and also satisfying an integral constraint of the form

$$J_2 \equiv f_2\ (y_i, y_f) + \int_{x_i}^{x_f} \phi_2\ (y,\ \dot{y})\ dx \tag{2}$$

Then according to the usual practice, the quantity which is to be extremised is

$$J \equiv J_1 + \lambda J_2 \equiv f(y_i, y_f) + \int_{x_i}^{x_f} \phi(y, \dot{y})\, dx \qquad (3)$$

where

$$f = f_1 + \lambda f_2$$
$$\phi = \phi_1 + \lambda \phi_2$$

and λ is a variable Lagrange multiplier.

Since the problem is of Bolza type in the calculus of variations, we accordingly write the first variation of J which is given by

$$\delta J \equiv (f y_i\ \delta y_i + f y_f\ \delta y_f) + \left[(\phi - \dot{y}\, \phi_{\dot{y}})\ \delta x + \phi_{\dot{y}}\ \delta y \right]_{x_i}^{x_f} + \int_{x_i}^{x_f} (\phi_y - \frac{d}{dx} \phi_{\dot{y}})\, \delta y\, dx \qquad (4)$$

For extremum $\delta J = 0$ and from the familiar arguments concerning the arbitrary variations, this equation leads to the following conditions:

(i) $\phi_y - \dfrac{d}{dx} \phi_{\dot{y}} = 0$ (Euler Equation)

In case the function ϕ is such that it does not contain the variable x explicitly in it then on integration the first integral of the Euler equation is obtained as

$$\dot{y}\ \phi_{\dot{y}} - \phi(y, \dot{y}) = C$$

where C is an integral constant.

(ii) The condition which must be satisfied at the initial point

$$f y_i - (\phi_{\dot{y}})_{x_i} = 0$$

(iii) The condition which must be satisfied at the final point

(a) $f y_f + (\phi_{\dot{y}})_{x_f} = 0$ { when the final values of y is not prescribed

(b) $(\phi - \dot{y}\, \phi_{\dot{y}})_{x_f} = 0$ { when the final value of x is not prescribed

(iv) In addition to the above conditions, also the Legendre's condition of the problem must always hold at every point of the optimum arc, ie,

$$\phi_{\dot{y}\dot{y}} \geqslant 0$$

FORMULATION OF THE EXACT PROBLEM

Having given a broad outline of the general problem of the calculus of variations under which we formulate our problem of finding the optimum shape of the body in free molecular flow whose drag is minimum, we now define our exact problem. The drag function, under the assumptions defined and also assuming that the body has a flat nose of radius y_i and the drag due to the base is neglected, is given by

$$\frac{D}{2\pi q} = k y_i^2 + y_f^2 + 2k \int_{x_i}^{x_f} \frac{y \dot{y}^3}{(1 + \dot{y}^2)^{\frac{3}{2}}}\, dx \qquad (5)$$

where q denotes the free-stream dynamic pressure, x the axial distance, $y(x)$ the radius of the body, $\dot{y}$ the derivative dy/dx and k a constant dependent upon the temperature ratio and the speed ratio. The subscripts i and f refer respectively to the initial and final points. Under the scheme given in the general formulations, the problem under consideration should have any one of the following two integral constraints :

Case A :—The surface area of the body is known to be a given quantity, ie

$$\frac{S}{2\pi} = \int_{x_i}^{x_f} y\,(1+\dot{y}^2)^{\frac{1}{2}}\,dx + \frac{y_i^2}{2} \tag{6}$$

Case B :—The volume of the body is known to be a prescribed quantity, ie,

$$\frac{V}{\pi} = \int_{x_i}^{x_f} y^2\,dx \tag{7}$$

Also if it is assumed that, in addition, either the length l or the base diameter t is prescribed then the minimal problem is to find in the class of arcs $y(x)$, the one which satisfies either (6) or (7) and is such that the functional (5) is a minimum. Obviously the defined problem falls into the category of the general formulation of the problem given earlier.

SOLUTION OF THE PROBLEM

Case A :—In this case

$$f \equiv (k + \lambda/2)\; y_i^2 + y_f^2 \tag{8}$$

$$\phi \equiv \frac{2k\, y\dot{y}^2}{(1+\dot{y}^2)^{\frac{1}{2}}} + \lambda\, y\sqrt{1+\dot{y}^2} \tag{9}$$

Therefore the functions f and ϕ must satisfy the four conditions given earlier.

The first integral is here given by

$$\frac{2k\, y\dot{y}^2}{(1+\dot{y}^2)^{3/2}} - \frac{\lambda\, y}{(1+\dot{y}^2)^{\frac{1}{2}}} = C \tag{10}$$

Now in this particular class of problem, there can be further two sub-classes, ie, (a) the case when the length of the body is free and the diameter is known and (b) the case when the diameter is free but the total length is known.

We first consider the situations (a) and from the condition (ii) and (iiib), we have

$$2k\, y_i\, \dot{y}_i\, \frac{2+\dot{y}_i^2}{(1+\dot{y}_i^2)^{3/2}} + \frac{\lambda\, y_i\, \dot{y}_i}{(1+\dot{y}_i^2)^{\frac{1}{2}}} - 2\,(k+\lambda/2)\, y_i = 0 \tag{11}$$

$$C = 0 \tag{12}$$

Also the condition (iv) gives

$$\dot{y}^2 \leqslant \frac{4\,k+\lambda}{2\,k-\lambda} \tag{13}$$

Now if we examine (10), (11) and (12), we see that the optimum curve passes through the origin, ie, the body has a sharp nose. To obtain the equation of the curve defining this body we must integrate the first integral under the conditions that $x_i = 0$, $y_i = 0$. We therefore obtain

$$y = \left[\frac{\lambda}{2\,k-\lambda}\right]^{\frac{1}{2}} x \tag{14}$$

Thus the optimum body under the conditions that the surface area and the diameter of the body are known quantities is conical in shape passing through the origin. This equation contains the Lagrange constant λ whose value is determined from the integral constraint and is given by

$$\lambda = \frac{\pi^2 t^4}{8 S^2} k \tag{15}$$

Therefore the equation of the body becomes

$$y = \left(\frac{16 S^2}{\pi^2 t^4} - 1 \right)^{\frac{1}{2}} x \tag{16}$$

Here we observe that the optimum body shape is independent of the parameter k, ie independent of the temperature ratio and the speed ratio. Once the body shape is known it is easy to find the drag by using (5). In case we define the drag coefficient $C_D = \frac{2 D}{\pi q t^2}$, we obtain the following result

$$C_D = 1 + \frac{4 k A^3}{(1 + A^2)^{\frac{1}{2}}} \cdot \frac{1}{\tau^2} \tag{17}$$

where τ is the thickness ratio defined as

$$\tau = t/l \tag{18}$$

and

$$A = \left(\frac{16 S^2}{\pi^2 t^4} - 1 \right)^{-\frac{1}{2}} \tag{19}$$

We thus see that the drag of the optimum body is linearly dependent upon the value of k. The relation between $\frac{4 S}{\pi t^2}$ and C_D is represented in Fig. 1 for three different values of k.

From (14), (18) and (19) we see that the optimum thickness ratio is given by

$$\tau = 2A \tag{20}$$

Again in the situation (b) when the length of the body is prescribed the first integral, the conditions (ii) and (iiia) are respectively represented by

$$\frac{2k\, y\, \dot{y}^2}{(1+\dot{y}^2)^{3/2}} - \frac{\lambda\, y}{(1+\dot{y}^2)^{\frac{1}{2}}} = C \tag{21}$$

$$2\,(k+\lambda/2)\, y_i - 2k\, y_i\, \dot{y}_i\, \frac{2+\dot{y}_i^2}{(1+\dot{y}_i^2)^{3/2}} - \frac{\lambda\, y_i\, \dot{y}_i^2}{(1+\dot{y}_i^2)^{\frac{1}{2}}} = 0 \tag{22}$$

$$2\, k\, y_f\, \dot{y}_f\, \frac{2+\dot{y}_f^2}{(1+\dot{y}_f^2)^{3/2}} + \frac{\lambda\, y_f\, \dot{y}_f}{(1+\dot{y}_f^2)^{\frac{1}{2}}} + 2 y_f = 0 \tag{23}$$

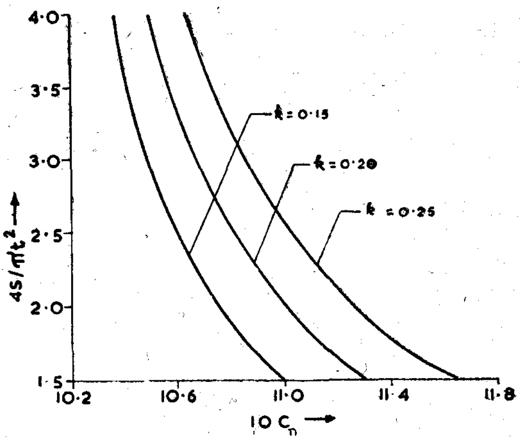

Fig. 1—Drag coefficient for given values of k.

If we examine (21) and (22), we observe that there is no possibility of the optimum curve passing through the origin and hence the optimum body is blunt. Here also the

Legendre's condition (13) holds and as a consequence of that at every point $(\lambda + k) < 0$. The values of $\dot{y}_f$ and $\dot{y}_i$ are obtained from

$$Y^3_f (4 - 4k^2 - \lambda^2 - 4\lambda k) + 2Y^2_f (6 - 8k^2 - \lambda^2 - 6\lambda k) + \\ + Y_f (12 - 16k^2 - \lambda^2 - 8\lambda k) + 4 = 0 \qquad (24)$$

$$Y_i^2 (\lambda^2 - 4k^2) + 2Y_i (\lambda^2 - 2k^2 - 2\lambda k) + 4 (k + \lambda/2)^2 = 0 \qquad (25)$$

where

$$Y_f = \dot{y}_f^2$$
$$Y_i = \dot{y}_i^2$$

The solution of these equations should be such that

$$Y_i < 1 , Y_f < 1$$

Now the first integral (21) gives

$$y = \frac{C (1 + \dot{y}^2)^{3/2}}{(2k - \lambda) \dot{y}^2 - \lambda} \qquad (26)$$

Since

$$x = \int \frac{dy}{\dot{y}}$$

the relation (26) leads to

$$x = C \int (1 + \dot{y}^2)^{\frac{1}{2}} \left\{ \frac{(2k - \lambda) \dot{y}^2 - (\lambda + 4k)}{[(2k - \lambda) \dot{y}^2 - \lambda]^2} \right\} d\dot{y} \\ = C F (\lambda, \dot{y}, k) + E \qquad (27)$$

Equations (26) and (27) are the parametric representation of the optimising curve. Also from (26) we obtain the initial and final values of the radius in the form

$$y_i = y_i (c, \lambda, k) \qquad (28)$$
$$y_f = y_f (c, \lambda, k) \qquad (29)$$

Also (27) gives

$$0 = C \{ F (\lambda, \dot{y}_i, k) \} + E \qquad (30)$$
$$l = C \{ F (\lambda, \dot{y}_f, k) \} + E \qquad (31)$$

Relations (30) and (31) determine the value of the constant C in terms of λ. To find the value of λ we make use of the integral constraint that the surface area of the body is a prescribed quantity. Knowing λ we know the values of y_i and y_f from (28) and (29) respectively.

Now if here we define the drag coefficient C_D as $D/2\pi q l^2$, then from (5) we can deduce that

$$C_D = (k - \lambda/2) (y_i/l)^2 + \frac{\tau^2}{4} + \frac{\lambda S}{2\pi l^2} + \frac{C}{l} \left[1 + \frac{1}{l} \int_0^l \dot{y}^2 \, dx \right] \qquad (32)$$

Case B :—In this case

$$\phi \equiv \frac{2 k y \dot{y}^2}{(1 + \dot{y}^2)^{\frac{1}{2}}} + \lambda y^2 \qquad (33)$$

$$f \equiv k y_i^2 + y_f^2 \qquad (34)$$

Therefore the first integral of the Euler equation is given by

$$\frac{2\,k\,y\,\dot{y}^2}{(1+\dot{y}^2)^{3/2}} - \lambda\,y^2 = C \tag{35}$$

and the Legendre's condition reduces to

$$2\,ky\,\frac{2-\dot{y}^2}{(1+\dot{y}^2)^{5/2}} \geqslant 0 \tag{36}$$

which shows that at every point along the optimal arc $\dot{y}^2 \leqslant 2$. As in Case A, here also two different situations can arise, (a) when the diameter of the body is specified but the length is free and (b) when the length is prescribed but the diameter is free.

In situation (a), the conditions (ii) and (iiib) give that

$$2\,k\,y_i - 2\,k\,y_i\,\frac{2\,\dot{y}_i + \dot{y}_i^3}{(1+\dot{y}_i^2)^{3/2}} = 0 \tag{37}$$

$$C = 0 \tag{38}$$

On examination of (33) and (38), we notice that the optimal curve passes through the origin and hence the minimum drag body is sharp nosed. Also the first integral can now be written as

$$\frac{2\,k\,\dot{y}^2}{(1+\dot{y}^2)^{3/2}} - \lambda\,y = 0 \tag{39}$$

which is to be solved under the given boundary conditions that $x_i = y_i = 0$; $y_f = t/2$. Also this equation shows that $\dot{y}_i = 0$ which means that the optimum body has a cusp at the origin. Again

$$y = \frac{2\,k}{\lambda}\;\frac{\dot{y}^2}{(1+\dot{y}^2)^{3/2}} \tag{40}$$

therefore

$$x = \frac{2\,k}{\lambda}\int\frac{(2-\dot{y}^2)}{(1+\dot{y}^2)^{5/2}}\,d\dot{y} \tag{41}$$

Integrating and applying the condition that at the origin $\dot{y} = 0$, we obtain

$$x = \frac{2\,k}{\lambda}\times\frac{\dot{y}\,(2+\dot{y}^2)}{(1+\dot{y}^2)^{3/2}} \tag{42}$$

Equations (40) and (42) act as the parametric representation of the optimising curve. The value of the Lagrange parameter λ is obtained as follows:

Since the diameter is known, we have from (40) that

$$\lambda t = 4k\,\frac{\dot{y}_f^2}{(1+\dot{y}_f^2)^{3/2}} \tag{43}$$

On solving this equation for $\dot{y}_f$, we use it to solve the integral

$$V = \pi\left(\frac{2k}{\lambda}\right)\int_0^{\dot{y}_f}\frac{\dot{y}^4}{(1+\dot{y}^2)^3}\;\frac{dx}{d\dot{y}}\,d\dot{y} \tag{44}$$

which gives the value of λ for given value of V. Now if we define the drag coefficient a $2D/\pi q t^2$, we have in this case

$$C_D = 1 + \frac{4\lambda}{t^2}\left[\frac{V}{\pi} + \int_0^l y^2 \dot{y}^2\, dx\right] \tag{45}$$

In situation (b), when the length is prescribed but the final diameter is free, we have the first integral and the conditions (ii) and (iii a) applicable and we thus obtain

$$\frac{2\,k\,y\,\dot{y}^2}{(1+\dot{y}^2)^{3/2}} - \lambda y^2 = C \tag{46}$$

$$2\,k y_i\left[1 - \frac{2\dot{y}_i + \dot{y}_i^3}{(1+\dot{y}_i^2)^{3/2}}\right] = 0 \tag{47}$$

$$2\,y_f\left[1 + k\,\frac{2\,\dot{y}_f + \dot{y}_i^3}{(1+\dot{y}_f^2)^{3/2}}\right] = 0 \tag{48}$$

An examination of these relations shows that the optimum arc cannot pass through the origin and that the body is flat nosed. Also the Legendre's condition (36) holds, ie at every point of the extremal arc $\dot{y}^2 \leqslant 2$. The equation (47) shows that $\dot{y}_i^2 = \frac{-1+\sqrt{5}}{2}$ and $\dot{y}_f$ is obtained from (48). Therefore from (46), we have

$$C = y_i\,(B - \lambda y_i) \tag{49}$$

where

$$B = k\,\frac{(-1+\sqrt{5})}{\left(1+\frac{-1+\sqrt{5}}{2}\right)^{3/2}}$$

Again (46) may be rewritten as

$$\lambda y = \alpha + [\alpha^2 - \lambda y_i\,(B - \lambda y_i)]^{\frac{1}{2}} \tag{50}$$

where

$$\alpha = \frac{k\dot{y}^2}{(1+\dot{y}^2)^{3/2}}$$

Therefore

$$\lambda x = \int_{\dot{y}_i}^{\dot{y}_f} \frac{1}{\dot{y}}\,\frac{d\,(\lambda y)}{d\dot{y}}\,d\dot{y} \tag{51}$$

Thus (50) and (51) represent the parametric equation of the optimum body.

Again, since the volume of the body is supposed to be given, we have

$$\lambda^3 V = \pi k \int_{\dot{y}_i}^{\dot{y}_f} \left\{\alpha + \left[\alpha^2 - \lambda y_i\,(B - \lambda y_i)\right]^{\frac{1}{2}}\right\}\left\{1 + \frac{\alpha}{[\,\alpha^2 - \lambda y_i\,(B - \lambda y_i)\,]^{\frac{1}{2}}}\right\}\beta\, d\dot{y} \tag{52}$$

where

$$\beta = \frac{2-\dot{y}^2}{(1+\dot{y}^2)^{5/2}}$$

Numerical solution of the problem can be obtained by adopting the following procedure:

Step 1—Assume an appropriate value for λy_i.

Step 2—Integrate (51) and since l is known, we obtain the value of λ.

Step 3—Using this value of λ, calculate V from (52).

Step 4—If the value of V so calculated is not equal to the specified value then the value of λy_i is further adjusted to satisfy the volume constraint (52).

Step 5—Having calculated the correct value of λ the body shape is to be computed by the simultaneous solution of (50) and (51).

Also, in this case, if we define the drag coefficient C_D as $D/2\pi q l^2$, then with the help of (5), we have

$$C_D = \frac{\tau^2}{4} + \frac{1}{l^2} \left[k y_i^2 + \lambda \frac{V}{\pi} + Cl + \lambda \int_0^l y^2 \dot{y}^2 \, dx + C \int_0^l \dot{y}^2 \, dx \right. \tag{53}$$

CONCLUSIONS

The problem of determining the optimum non-slender body of revolution in free molecular flow under the integral constraint of the surface area or the volume has been solved for the situations when either the length of the body or the final diameter of the body is a known quantity. From the above general analysis we can make the following few general observations :

(i) When the length is free and the final diameter of the body is given, the optimum body is sharp nosed irrespective of the fact whether the surface area or the volume of the body is the specified integral constraint; and when the length is known and the final diameter is free, the optimum body is blunt nosed.

(ii) Also when the length is free and final diameter is given and the surface area is the specified integral constraint then the optimum body does not depend upon the value of the constant k but the drag coefficient does depend on the value of k.

(iii) When the final diameter of the body is a known quantity but the length is free and the integral constraint of the problem is that the volume of the body is a given quantity then the optimum body has a cusp at the origin.

ACKNOWLEDGEMENTS

The authors are grateful to Dr. R. R. Aggarwal, PScO for his interest in this work and to the Director, Defence Science Laboratory, Delhi for his permission to publish this paper.

REFERENCES

1. MIELE, A., *J. Franklin Inst*, **283** (1967), 169.
2. CARTER, W. J., *J. Aerospace Sci*, **24** (1971), 527.
3. TAN, H. S., *J. Aerospace Sci*, **25** (1958), 56.
4. CHANG, I. D., *J. Aerospace Sci*, **25** (1958), 57.
5. TAN, H. S., *J. Aerospace Sci*, **25** (1958), 263.
6. TAN, H. S., *Quart Appl Maths*, **17** (1959), 311.
7. TAN, H. S., *J. Aerospace Sci*, **26** (1959), 360.
8. MIELE, A., "Theory of Optimum Aerodynamic Shapes", (Academic Press, New York), (1965), 415.
9. TAN, H. S., "Theory of Optimum Aerodynamic Shapes", (Academic Press, New York), (1965), 423.