EXPANSION OF KAMPÉ DE FÉRIET FUNCTION IN SERIES OF OTHER KAMPÉ DE FÉRIET FUNCTIONS

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In this paper an expansion formula for the Kampé de Fériet function has been given in series of other such functions.

Luke & Coleman[1] have given expansion formula for a hypergeometric function in series of other hypergeometric functions. Field & Luke[2] have generalized this formula by Laplace transform techniques. Wimp & Luke[3] have generalized this result of Field & Luke[2] further to include a multiplier of a power of independent variable. In this paper we have established such an expansion theorem for the Kampé de Fériet function by using the result given by Wimp & Luke[3]. For the sake of simplicity we represent the hypergeometric function of two variables given by Kampé de Fériet[4] as

$$F\begin{matrix} g:h\,;H \\ i:j\,;\,J\end{matrix}\left[\begin{matrix} a_g : b_h\,;B_H\,; \\ \alpha_i : c_j\,;C_J\,;\end{matrix} x,y\right] = \sum\sum_{p,q=0}^{\infty} \frac{\Pi(a_g)_{p+q}\,\Pi(b_h)_p\,\Pi(B_H)_q\,x^p\,y^q}{\Pi(\alpha_i)_{p+q}\,\Pi(c_j)_p\,\Pi(C_J)_q\,p!\,q!} \quad (1)$$

where $(\mu)_p$ stands for the symbol $\Gamma(\mu+p)\,/\,\Gamma(p)$, μ_i denotes the set of parameters $\mu_1, \mu_2, \ldots, \mu_i$. Colon $(:)$ and semicolon $(;)$ separate the terms of the form $(\alpha_i)_{p+q}$, $(c_j)_p$, $(C_J)_q$ etc.

Ragab[5] has shown that the series on the right hand side of (1) converges for all finite x, y if $g+h \leqslant i+j$, and $g+H \leqslant i+J$, converges for $|x| + |y| < \min(1, 2^{i-g+1})$ if $g+h=i+j+1$, and $g+H = i+J+1$. It can easily be shown that the series converges if

$$g+h = i+j+1, \quad g+H = i+J+1$$

and

$$|x| \leqslant 1\,,\ |y| < 1\,,\ \mathrm{Re}(X) > 0\,,$$

or

$$|x| < 1\,,\ |y| \leqslant 1\,,\ \mathrm{Re}(Y) > 0\,,$$

or

$$|x| \leqslant 1\,,\ |y| \leqslant 1\,,\ \mathrm{Re}(X) > 0\,,\ \mathrm{Re}(Y) > 0\,,$$

where

$$X = \Sigma\,\alpha_i + \Sigma\,c_j - \Sigma\,a_g - \Sigma\,b_h; \quad Y = \Sigma\,\alpha_i + \Sigma\,C_J - \Sigma\,a_g - \Sigma\,B_H\,.$$

Luke & Coleman[1] have given an expansion formula for a hypergeometric function in series of other hypergeometric functions. Field & Wimp[2] have generalized this formula by Laplace transform techniques. Wimp & Luke[3] have generalized this result[2] further, to include a multiplier of a power of the independent variable by establishing the following theorem :

Theorem

$$\left.\begin{array}{l}
\text{(i) Let none of the following be negative integers} \\
\quad \mu;\ \alpha_t+\mu-1;\ c_r-\mu-1;\ \gamma;\ \beta_u-1 . \\
\text{(ii) Let } p, q, r \text{ and } s \text{ be positive integers or zero and} \\
\quad p+r \leqslant q+s \quad \text{or } q+s+1 = p+r \text{ if } |zw| < 1 , \\
\quad p+t \leqslant q+u+1 \text{ or } q+u+2 = p+t \text{ if } |z| < 1 , \\
\quad r+u+1 = s+t . \\
\text{(iii) Let } 0 < w < 1 . \\
\text{(iv) Let the following identities be satisfied :} \\
\qquad \mu(s-r-2)+\Sigma c_r+\Sigma \beta_u-\Sigma \alpha_t-\Sigma d_s < 1/2 ; \\
\qquad c_r > 0,\ \beta_u+\mu > 0 .
\end{array}\right\} \tag{2}$$

Then

$$w^{\mu}\ {}_{p+r}F_{q+s}\,(a_p\,,\,c_r\,;\,b_q\,,\,d_s\,;\,zw)$$

$$= \frac{\Pi(c_r)_{-\mu}\ \Pi(\alpha_t)_{\mu}}{\Pi(d_s)_{-\mu}\ \Pi(\beta_u)_{\mu}} \sum_{n=0}^{\infty} \frac{(2n+\gamma)(-\mu)_n}{n!\,(n+\gamma)_{\mu+1}} \,.$$

$$\cdot\,{}_{p+t+1}F_{q+u+2}\left[\begin{matrix} \mu+1,\ \alpha_t+\mu,\ a_p\,; \\ +\mu-n+1,\ n+\gamma+\mu+1,\ \beta_u+\mu,\ b_q\,; \end{matrix}\ z\right] .$$

$$\cdot\,{}_{r+u+2}F_{s+t}\left[\begin{matrix} -n,\ n+\gamma,\ c_r-\mu,\ \beta_u\,; \\ a_t,\ d_s-\mu\,; \end{matrix}\ w\right] \tag{3}$$

Authors[3] further proved that (3) is defined and converges under stated hypothesis (2).

By employing the above theorem, we shall establish the following theorem :

THEOREM

$$\left.\begin{array}{l}
\text{(i) Let none of the following be negative integers} \\
\quad \text{(a) } \rho;\ \gamma;\ \theta_u-1;\ \beta_t+\rho-1;\ B_H-\rho-1 ; \\
\quad \text{(b) } w;\ \delta;\ \psi_v-1;\ \phi_s+w-1;\ b_h-w-1 . \\
\text{(ii) Let } g, h, H, i, j, J \text{ be positive integers or zero and} \\
\quad \text{(a) } g+H \leqslant i+J,\quad g+t \leqslant i+u+1,\quad J+t \leqslant H+u+1 , \\
\quad \text{(b) } g+h \leqslant i+j,\quad g+s \leqslant i+v+1,\quad J+s \leqslant H+v+1 . \\
\text{(iii) Let} \\
\quad \text{(a) } 0 < \mu < 1 , \\
\quad \text{(b) } 0 < \lambda < 1 . \\
\text{(iv) Let the following inequalities be satisfied :} \\
\quad \text{(a) } \rho(J-H-2)+\Sigma B_H+\Sigma \theta_u-\Sigma C_J-\Sigma \beta_t < 1/2;\quad B_H > 0,\ \theta_u+\rho > 0 , \\
\quad \text{(b) } w(j-h-2)+\Sigma b_h+\Sigma \psi_v-\Sigma c_j-\Sigma \phi_s < 1/2;\quad b_h > 0,\ \psi_v+u > 0 .
\end{array}\right\} \tag{4}$$

Then

$$F = \lambda^w \mu^\rho F^{g:h;H}_{i:j;J}\left[\begin{matrix} a_g: b_h; B_H; \\ \alpha_i: c_j; C_J; \end{matrix} \lambda x, \mu y\right]$$

$$= \frac{\Pi(b_h)_{-w}\ \Pi(B_H)_{-\rho}\ \Pi(\beta_t)_\rho\ \Pi(\phi_s)_w}{\Pi(c_j)_{-w}\ \Pi(C_J)_{-\rho}\ \Pi(\theta_u)_\rho\ \Pi(\psi_v)_\rho} \cdot \sum_{m,n=0}^{\infty}\sum \frac{(2m+\delta)\ (2n+\gamma)\ (-\rho)_n\ (-w)_n}{m!\ n!\ (n+\gamma)_{\rho+1}\ (m+\delta)_{w+1}} \cdot$$

$$\cdot F^{g:s+1;t+1}_{i:v+2;u+2}\left[\begin{matrix} a_g: \phi_s+w,\ w+1;\ \beta_u+\rho,\ \rho+1; \\ \alpha_i: w-m+1,\ w+\delta+m+1,\ \psi_v+w;\ \rho-n+1,\ \rho+\gamma+n+1,\ \theta_u+\rho; \end{matrix} x, y\right] \cdot$$

$$\cdot\ {}_{h+v+2}F_{j+s}(-m, m+\delta,\ b_h-w,\ \psi_v;\ \phi_s,\ C_j-w;\ \lambda)\ .$$

$$\cdot\ {}_{H+u+2}G_{J+t}(-n, n+\gamma,\ B_H-\rho,\ \theta_u;\ \beta_t,\ C_J-\rho;\ \mu) \qquad (5)$$

Proof : Expressing the Kampé de Fériet function in (5) as in (1) we have

$$F = \lambda^w \mu^\rho \sum_{p,q=0}^{\infty}\sum \frac{\Pi(a_g)_{p+q}\ \Pi(b_h)_p\ \Pi(B_H)_q\ (\lambda x)^p\ (\mu y)^q}{\Pi(a_i)_{p+q}\ \Pi(c_j)_p\ \Pi(C_J)_q\ p!\ q!}$$

$$= \lambda^w \sum_{p=0}^{\infty} \frac{\Pi(a_g)_p\ \Pi(b_h)_p}{\Pi(\alpha_i)_p\ \Pi(c_j)_p} \frac{(\lambda x)^p}{p!} \left\{ \mu^p\ {}_{g+H}F_{i+J}(a_g+p, B_H;\ \alpha_i+p, C_J;\ \mu y) \right\}$$

Now evaluating the expression in parenthesis with the help of (3) which is defined and converges under the conditions (*a*) of the hypothesis (4), we have

$$F = \lambda^w \sum_{p=0}^{\infty} \frac{\Pi(a_g)_p\ \Pi(b_h)_p\ (\lambda x)^p}{\Pi(\alpha_i)_p\ \Pi(c_j)_p\ p!} \frac{\Pi(B_H)_{-\rho}\ \Pi(\beta_t)_\rho}{\Pi(\theta_u)_\rho\ \Pi(C_J)_{-\rho}} \sum_{n=0}^{\infty} \frac{(2n+\gamma)\ (-\rho)_n}{n!\ (n+\gamma)_{\rho+1}} \cdot$$

$$\cdot\ {}_{g+t+1}F_{i+u+2}(\rho+1,\ \beta_t+\rho,\ a_g+p;\ \rho-n+1,\ n+\gamma+\rho+1,\ \alpha_i+p;\ y)\ \cdot$$

$$\cdot\ {}_{H+u+2}F_{J+t}(-n,\ n+\gamma,\ B_H-\rho,\ \theta_u;\ \beta_t,\ C_J-\rho;\ \mu)$$

$$= \frac{\Pi(B_H)_{-\rho}\ \Pi(\beta_t)_\rho}{\Pi(C_J)_{-\rho}\ \Pi(\theta_u)_\rho} \sum_{n=0}^{\infty} \frac{(2n+\gamma)\ (-\rho)_n}{n!\ (n+\gamma)_{\rho+1}} \left\{ \lambda^w F^{g:t+1;h}_{i:u+2;j} \cdot \right.$$

$$\left. \cdot \left[\begin{matrix} a_g: \beta_t+\rho,\ \rho+1;\ b_h; \\ \alpha_i: \theta_u+\rho,\ \rho+\gamma+n+1,\ \rho-n+1;\ C_j; \end{matrix} y, \lambda x\right]\right\}$$

$$\cdot\ {}_{H+u+2}F_{J+t}(-n,\ n+\gamma,\ B_H-\rho,\ \theta_u;\ \beta_t,\ C_J-\rho;\ \mu)$$

Repeating this procedure once again for the expression in the parenthesis, we have (5).

We can derive a further result as a confluent form of the above theorem which follows upon using

$$\lim_{a\to\infty} (a)_n\ (z/a)^n = z^n$$

and, in (5), replacing λ by λ/δ and x by δx ; μ by μ/γ and y by γy and letting $\to\infty,\ \mu\to\infty$.

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