APPROXIMATE ANALYTIC SOLUTIONS FOR ONE-DIMENSIONAL PISTON PROBLEM WITH RADIATIVE TRANSFER

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We have considered the motion of a spherical, cylindrical or plane piston into a non-uniform medium taking into account energy transfer by radiation. We have obtained approximate analytic solutions for the physical situation (*i*) when the medium is optically thin and (*ii*) when it is optically thick. We have made use of technique due to Chernyi in which the flow variables are expanded in series of powers of ϵ, the density ratio across the shock. Temperature distribution behind the shock front is depicted graphically for the limiting cases, mentioned above for a large number of cases including the situations when the medium ahead of the shock is homogeneous or otherwise.

The equations governing radiation-gas-dynamics being highly non-linear integro-differential or differential in different approximations such as the modified form of Schuster-Schwarzschild's approximation[1], optically thick approximation and optically thin approximation, the solutions have mainly been obtained by numerically integrating the differential equations in similarity variables [2–4]. Some attempts at finding approximate analytic solutions have recently been made by Traugott[5], Wang[6] and Marshak[7]. Wang[6] has obtained, employing Chernyi's technique[8], some particular approximate analytic solutions upto the zeroth order only, for the plane piston problem in an inhomogeneous medium for the optically thin case and the 'local temperature approximation', generally in terms of a similarity variable and for some special opacity laws. As the previous numerical results[4,9] showed that the radiative transfer affects mostly density and temperature distribution, while the pressure and velocity remain practically unaffected; the analysis in the Chernyi's method was simplified by the introduction of the assumption that the velocity and pressure behind the shock are the same as in the non-radiating case. Wang[6] has assumed the plane piston (causing motion) to be cool.

We consider the approximate analytic solutions for the piston problem with thermal radiation more generally, that is, when the piston is spherical, cylindrical or plane. We also consider the cases when the piston has a finite temperature. We have treated the optically thin and thick cases for different opacity laws governing different ranges of temperatures. We have also employed Chernyi's method[8] under the same assumptions as made by Wang[6]. We have obtained the solutions for the optically thin case upto first order, zeroth order solution being in a closed form while the first order solution is generally expressible in terms of incomplete Beta functions which, however, can be integrated in a closed form for special choices of the parameters. For the optically thick case, the zeroth order solution for temperature is governed by the general form of diffusion equation. By introducing a similarity variable, the solution of this equation is obtained in terms of confluent hypergeometric functions for a particular choice of the exponents α and β in the opacity law. The first order term in this case does not seem to be amenable to a closed form solution.

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BASIC EQUATIONS

The relevent equations in terms of time t and the Lagrangian co-ordinate η are;

Equation of continuity :

$$\frac{\partial x}{\partial \eta} = \frac{1}{\rho x^{j-1}} \tag{1}$$

Equation of momentum :

$$\frac{\partial^2 x}{\partial t^2} = - x^{j-1} \frac{\partial p}{\partial \eta} \tag{2}$$

Equation of energy :

$$\frac{\partial e}{\partial t} + p \frac{\partial \rho^{-1}}{\partial t} = - \frac{1}{\rho} \nabla . \vec{F}^R \tag{3}$$

$$e = c_v T \tag{4}$$

$$\frac{d\tau}{dx} = K \tag{5}$$

where p is pressure, ρ density, T temperature, e internal energy, $\vec{F}^R$ is the radiative energy flux vector, x the one-dimensional space co-ordinate, K the absorption coefficient, τ the optical thickness, $j=1,2,3$, for plane, cylindrical and spherical symmetry respectively. η is the Lagrangian co-ordinate defined by $d\eta = \rho_0 \, x^{j-1} \, dx_0$, where x_0 is the value of x at the initial instant of time and $\rho_0 = A \, x_0^{-\omega}$ is the initial density and ω is a positive constant.

Following Chernyi[8] we assume the following expansions for x, p, ρ, T, e and $\vec{F}^R$

$$x = x^{(0)} + \epsilon \, x^{(1)} + \epsilon^2 \, x^{(2)} + \ldots\ldots \tag{6}$$

$$p = p^{(0)} + \epsilon \, p^{(1)} + \epsilon^2 \, p^{(2)} + \ldots\ldots \tag{7}$$

$$\rho = \rho^{(0)}/\epsilon + \rho^{(1)} + \epsilon \, \rho^{(2)} + \ldots\ldots \tag{8}$$

$$T = T^{(0)} + \epsilon \, T^{(1)} + \epsilon^2 \, T^{(2)} + \ldots\ldots \tag{9}$$

$$e = e^{(0)} + \epsilon \, e^{(1)} + \epsilon^2 \, e^{(2)} + \ldots\ldots \tag{10}$$

$$\vec{F}^R = \vec{F}^{R(0)} + \epsilon \, \vec{F}^{R(1)} + \epsilon^2 \, \vec{F}^{R(2)} + \ldots \tag{11}$$

The problem is discussed in two media (*i*) optically thin medium and (*ii*) optically thick medium.

OPTICALLY THIN MEDIUM

We first consider the case when the medium is optically thin. Following Bloor[10], we take energy flux vector $\vec{F}^R$ and the absorption coefficient K as

$$\nabla . \vec{F}^R = C \, \rho^{\alpha} \, T^{\beta-4} \, (T^4 - C'^4) \tag{12}$$

$$K = k_1 \, \rho^{\alpha} \, T^{\beta-4} \tag{13}$$

where α, β, C, C' and k_1 are constants, C' being the temperature of the piston.

(*i*) We first consider the case when the piston is cool, that is, when $C' = 0$.

By using (12) and the gas law, (3) reduces to

$$\frac{\partial e}{\partial t} + p \frac{\partial \rho^{-1}}{\partial t} = - CR^{1-\alpha} p^{\alpha-1} T^{\beta-\alpha+1} \tag{14}$$

where R is the gas constant.

Substituting the expansions in (14), we get the following equations for the zeroth and first order terms.

$$\frac{\partial e^{(0)}}{\partial t} = - C R^{1-\alpha} {p^{(0)}}^{\alpha-1} {T^{(0)}}^{\beta-\alpha+1} \tag{15}$$

$$\frac{\partial e^{(1)}}{\partial t} - \frac{p^{(0)}}{\rho^{(0)2}} \frac{\partial \rho^{(0)}}{\partial t} = - C R^{1-\alpha} {p^{(0)}}^{\alpha-1} \times$$
$$\times {T^{(0)}}^{\beta-\alpha+1} \left[(\alpha-1) \frac{p^{(1)}}{p^{(0)}} + (\beta-\alpha+1) \frac{T^{(1)}}{T^{(0)}} \right] \tag{16}$$

The boundary conditions at the shock are

$$\left.\begin{aligned}
x^{(0)}(t) &= x_s \\
x_2^{(1)} &= x_2^{(2)} = \ldots\ldots = 0 \\
p_2^{(0)} &= \frac{2\rho_1}{\gamma+1} \dot{x}^{(0)2} \\
p_2^{(1)} &= p_2^{(2)} = \ldots\ldots = 0 \\
\rho_2^{(0)} &= \rho_1 \\
\rho_2^{(1)} &= \rho_2^{(2)} = \ldots\ldots = 0 \\
T_2^{(0)} &= L\, t^{2n} \\
T_2^{(1)} &= 0 \\
\eta_s &= \frac{A\, {x^{(0)}}^{j-\omega}}{j-\omega}
\end{aligned}\right\} \tag{17}$$

where A is a constant and $L = \dfrac{2\epsilon c_s^2 (n+1)^2}{(\gamma+1) R}$

Now we introduce a new variable t^* related to η through $\eta = \dfrac{A\, {x^{(0)}}^{j-\omega}(t^*)}{j-\omega}$, that is, t^* is the instant of time at which the shock crosses the particle with the Lagrangian co-ordinate η. We define the similarity variable μ by $\mu = \dfrac{\eta}{\eta_s}$ and this variable is related to t^* and t by

$$\mu = \left(\frac{t^*}{t}\right)^{(j-\omega)(n+1)} \tag{18}$$

Thus we can write

$$\frac{T_2^{(0)}(t^*)}{T_2^{(0)}(t)} = \left(\frac{t^*}{t}\right)^{2n} = \mu^{\frac{2n}{(j-\omega)(n+1)}} \tag{19}$$

Replacing $e^{(0)}$ by $c_v T^{(0)}$ and integrating (15) with boundary conditions (17), we get

$$T^{(0)}(t,\eta) = T_2^{(0)}(t^*)\left[1 + \frac{CR^{1-\alpha}(\beta-\alpha)}{c_v} T_2^{(0)\beta-\alpha}(t^*) \times \right.$$

$$\left. \times \int_{t^*}^{t} p^{(0)^{\alpha-1}}\, dt\right]^{-\frac{1}{\beta-\alpha}} \qquad (20)$$

For air at a temperature below 15,000° K, it has been generally agreed that the constants α and β can be approximately taken to be 1 and 9. In order that

$$\frac{T^{(0)}(t,\eta)}{T_2^{(0)}(t)}$$

may be a function of μ alone, when $\alpha = 1$, the similarity condition

$$2n\ (\beta - 1) + 1 = 0 \qquad (21)$$

must be satisfied. Under these conditions, (20) reduces to

$$\bar{T}^{(0)}(\mu) = \frac{T^{(0)}(t,\eta)}{T_2^{(0)}(t)}$$

$$= \mu^{\frac{2n}{(j-\omega)(n+1)}}\left[1 + \Gamma\left(\mu^{-\frac{1}{(j-\omega)(n+1)}} - 1\right)\right]^{-\frac{1}{\beta-1}}$$

$$= \left[(1-\Gamma)\,\mu^{\frac{1}{(j-\omega)(n+1)}} + \Gamma\right]^{-\frac{1}{\beta-1}} \qquad (22)$$

where the radiation parameter

$$\Gamma = \frac{C(\beta-1)(\gamma-1)}{R^{\beta}}\left\{\frac{2\epsilon}{\gamma+1}\right\}^{\beta-1}\left\{c_s(n+1)\right\}^{2(\beta-1)} \qquad (23)$$

For the zeroth order pressure $p^{(0)}$ we take the non-radiating solution[11] which is

$$p^{(0)} = \frac{2\rho_1 \dot{x}^{(0)^2}}{(\gamma+1)}\left[1 + k(1-\mu)\right] \qquad (24)$$

where $\quad k = \dfrac{(\gamma+1)\,n}{2(j-\alpha)(n+1)}$

The density $\rho^{(0)}$ is obtained from the gas law, (22) and (24) and is given by

$$\rho^{(0)} = \rho_1\left[1 + k(1-\mu)\right]\left[(1-\Gamma)\,\mu^{\frac{1}{(j-\omega)(n+1)}} + \Gamma\right]^{\frac{1}{\beta-1}} \qquad (25)$$

when $\alpha = 1$, (16) reduces to

$$\frac{\partial T^{(1)}}{\partial t} + \frac{C\beta}{c_v} T^{(0)^{\beta-1}}\, T^{(1)} = \frac{p^{(0)}}{c_v\, \rho^{(0)^2}}\,\frac{\partial \rho^{(0)}}{\partial t} \qquad (26)$$

Integrating (26), we get

$$\overline{T}^{(1)}(\mu) = \frac{T^{(1)}(t,\eta)}{T_2^{(0)}(t)} = (\gamma+1)\,\mu^m \Big[(1-\Gamma)\,\mu^m + \Gamma\Big]^{-\lambda} \times$$

$$\times \Big[A_1 \ln\mu + B_1 \ln(1+k-k\mu) + D(1-\mu^{-m}) -$$

$$- E\int_1^{\mu} \frac{\mu^{-m}}{1+k-k\mu}\, d\mu\Big] \qquad (27)$$

where

$$\left.\begin{aligned} m &= \frac{1}{(j-\omega)(n+1)}, \\ \lambda &= \frac{\beta}{\beta-1}, \\ A_1 &= \frac{(1-\Gamma)}{(j-\omega)}\left[\omega + \frac{1}{(n+1)(\beta-1)}\right], \\ B_1 &= (1-\Gamma), \\ D &= \Gamma\,\omega\,(n+1), \\ E &= k\,\Gamma. \end{aligned}\right\} \qquad (28)$$

The integral $\displaystyle\int_1^{\mu} \frac{\mu^{-m}}{1+k-k\mu}\, d\mu$

can be expressed in terms of incomplete Beta function as

$$\frac{k'^{(m-1)}}{(1-k')^m}\Big[B_\zeta(1-m,m) - B_{k'}(1-m,m)\Big] \qquad (29)$$

where $\quad k' = -k$ and $\zeta = \dfrac{k'}{k' + (1-k')\,\mu^{-1}}$

If $m = 1, 2, \ldots$ the incomplete Beta function can be integrated and we get the following solutions for $m = 1$ and 2.

(*a*) when $m = 1$,

$$\overline{T}^{(1)}(\mu) = (\gamma+1)\,\mu\Big[(1-\Gamma)\,\mu + \Gamma\Big]^{-\lambda} \times$$

$$\times \Big[\overline{A}\ln\mu + \overline{B}\ln(1+k-k\mu) + D(1-\mu^{-1})\Big] \qquad (30)$$

where $\quad \overline{A} = A_1 - \dfrac{E}{1+k}, \quad \overline{B} = B_1 + \dfrac{E}{1+k}.$

(b) when $m = 2$, the solution is given by

$$\overline{T}^{(1)}(\mu) = (\gamma+1)\,\mu^2 \left[(1-\Gamma)\,\mu^2 + \Gamma\right]^{-\lambda} \times$$
$$\times \left[\overline{A}_1 \ln \mu + \overline{B}_1 \ln(1+k-k\mu) + \overline{C}(\mu^{-1}-1) + D(1-\mu^{-2})\right] \quad (31)$$

where $\overline{A}_1 = A_1 - \dfrac{E\,k}{(1+k)^2}$, $\overline{B}_1 = B_1 + \dfrac{E\,k}{1+k^2}$, $\overline{C} = \dfrac{E}{1+k}$.

Substituting the expansions (6) and (8) in (1), we get

$$\frac{\partial x^{(1)}}{\partial \eta} = \frac{1}{\rho^{(0)}\, x^{(0)\,j-1}} \quad (32)$$

Integrating we get

$$x^{(1)} = x^{(0)} \int_1^{\mu} \frac{\left[(1-\Gamma)\,\mu^m + \Gamma\right]^{-\frac{1}{\beta-1}}}{[1+k-k\,\mu]}\, d\mu \quad (33)$$

The distance between the piston and the shock is given by

$$\frac{x_s - x_p}{x_s} = \epsilon \int_0^1 \frac{\left[(1-\Gamma)\,\mu^m + \Gamma\right]^{-\frac{1}{\beta-1}}}{[1+k-k\,\mu]}\, d\mu \quad (34)$$

(ii) Now we consider the case when the temperature at the piston is finite, that is, $C' \neq 0$.

From equations (3), (12) and the gas law, we get

$$\frac{\partial e}{\partial t} + p\,\frac{\partial \rho^{-1}}{\partial t} = -\,C\,R^{1-\alpha}\,p^{\alpha-1}\,T^{\beta-\alpha-3}\,(T^4 - C'^4) \quad (35)$$

where R is the gas constant.

Substituting the expansions (6) to (10) in (35), we get

$$\frac{\partial e^{(0)}}{\partial t} = -\,C\,R^{1-\alpha}\,{p^{(0)}}^{1-\alpha}\,{T^{(0)}}^{\beta-\alpha-3}\,({T^{(0)}}^4 - C'^4) \quad (36)$$

$$\frac{\partial e^{(1)}}{\partial t} - \frac{p^{(0)}}{{\rho^{(0)}}^2}\,\frac{\partial \rho^{(0)}}{\partial t} = -\,C\,R^{1-\alpha}\,{p^{(0)}}^{\alpha-1}\,{T^{(0)}}^{\beta-\alpha-3} \times$$
$$\times \left[{T^{(0)}}^4 \left\{(\alpha-1)\,\frac{p^{(1)}}{p^{(0)}} + (\beta-\alpha+1)\,\frac{T^{(1)}}{T^{(0)}}\right\} - C'^4 \left\{(\alpha-1) \times \right.\right.$$
$$\left.\left. \times\,\frac{p^{(1)}}{p^{(0)}} + (\beta-\alpha-3)\,\frac{T^{(1)}}{T^{(0)}}\right\}\right] \quad (37)$$

From (36) and (4), we obtain.

$$\frac{\partial T^{(0)}}{\partial t} = -\,\overline{k}\,{p^{(0)}}^{\alpha-1}\left[C{T^{(0)}}^{\alpha'} - \lambda_1\,{T^{(0)}}^{\beta'}\right] \quad (38)$$

where
$$\bar{k} = \frac{R^{1-\alpha}}{c_v}, \ \alpha' = \beta - \alpha + 1,$$
$$\beta' = \beta - \alpha - 3$$
$$\lambda_1 = C C'^4$$

By the transformation $z = T^{(0)^{1-\alpha'}}$, we get
$$\frac{\partial z}{\partial t} = -\bar{k}(1-\alpha')\, p^{(0)\alpha'-1} \left[C - \lambda_1 z^{\delta_1} \right] \tag{39}$$

where
$$\delta_1 = \frac{\beta' - \alpha'}{1-\alpha'} = \frac{-4}{\alpha - \beta}$$

We take $\alpha = 1$ and $\beta = 9$ then $\delta_1 = \frac{1}{2}$ and (39) integrates to
$$T^{(0)^{-4}} + \frac{C}{\lambda_1} \ln \left(C - \lambda_1 T^{(0)^{-4}}\right)$$
$$= T_2^{(0)^{-4}}(t^*) + \frac{C}{\lambda_1} \ln\{ C - \lambda_1 T_2^{(0)^{-4}}(t^*)\} - \frac{4\lambda_1}{c_v}(t - t^*) \tag{40}$$

Since $\dfrac{\lambda_1}{C\, T_2^{(0)^4}(t^*)} = \delta'^4 << 1$, where δ' is the ratio of temperature at the piston to that at the shock $T_2^{(0)^4}(t^*)$, we expand the logarithmic terms in (40) and obtain
$$T^{(0)} = T_2^{(0)}(t^*) \left[1 + \frac{8C(\gamma - 1)}{R} (t - t^*)\, T_2^{(0)8}(t^*) \right]^{-1/8} \tag{41}$$

We note that after cancelling certain terms the constant C' becomes a common factor on both sides of (41) and cancels out. We have retained terms up to $T^{(0)^{-8}}$. In order that the expression within the bracket on the right hand side of (41) may be a function of μ alone the similarity condition (21) must be satisfied and under this condition we get
$$\overline{T}^{(0)}(\mu) = \frac{T^{(0)}(t,\eta)}{T_2^{(0)}(t)} = \mu^{\frac{2n}{(j-\omega)(n+1)}} \left[1 + \Gamma\left(\mu^{-\frac{1}{(j-\omega)(n+1)}} - 1 \right) \right]^{-1/8}$$
$$= \left[(1-\Gamma)\, \mu^{\frac{1}{(j-\omega)(n+1)}} + \Gamma \right]^{-1/8} \tag{42}$$

where Γ is given by (24) with $\beta = 9$.

We assume the temperature at the piston to be a fraction of that at the shock so that
$$C' = \delta' T_2^{(0)}(t) \tag{43}$$

where δ' is a constant less than 1. Substituting (43) in (37) and putting $\alpha = 1$, $\beta = 9$ and $e^{(1)} = c_v T^{(1)}$, we get
$$\frac{\partial T^{(1)}}{\partial t} + \left\{ k_1 T^{(0)^8} - k_2 T_2^{(0)^8}(t) \right\} T^{(1)} = \frac{p^{(0)}}{c_v \rho^{(0)2}} \frac{\partial \rho^{(0)}}{\partial t} \tag{44}$$

where $$k_1 = \frac{9C}{c_v}\ ,\ k_2 = \frac{5C\,\delta'^4}{c_v}$$

Integrating (44), we get

$$\overline{T}^{(1)}\ (\mu) = \frac{T^{(1)}(t,\ \eta)}{T_2^{(0)}(t)}$$

$$= (\gamma + 1)\,\mu^{m(1-\lambda_3)}\left[(1-\Gamma)\,\mu^m + \Gamma\right]^{-\lambda_2} \times$$

$$\times\left[1 + (\theta_1\,\mu^m + 1)^{\frac{1}{2}}\right]^{2\lambda_3}\int_1^\mu\left[1 + (\theta_1\,\mu^m + 1)^{\frac{1}{2}}\right]^{-2\lambda_3}\mu^{m\lambda_3}\times$$

$$\times\left[\frac{A_{11}}{\mu} + B_{11}\,\mu^{-(m+1)} + \frac{C_1'}{1 + k - k\,\mu} + \frac{D_1}{\mu\,(1 + k - k\,\mu)}\right]d\mu \quad (45)$$

where
$$A_{11} = \frac{(1-\Gamma)\,\omega}{j-\omega}\ ,$$

$$B_{11} = \frac{\Gamma\,\omega}{j-\omega}\ ,$$

$$C_1 = -k\,(1-\Gamma)\left\{1 + \frac{m}{8}\right\},$$

$$D_1 = (1-\Gamma)\ (1+k)\,\frac{m}{8}\ ,$$

$$E_1 = -k\,\Gamma\ ,\ \theta_1 = \frac{1-\Gamma}{\Gamma}\ ,$$

$$\lambda_2 = \frac{k_1}{\Gamma}\left\{\frac{2\,\epsilon\,c_s^2\,(n+1)^2}{(\gamma+1)\,R}\right\}^8,$$

$$\lambda_3 = \frac{k_2}{\Gamma^{\frac{1}{2}}}\left\{\frac{2\,\epsilon\,C_s^2\,(n+1)^2}{(\gamma+1)\,R}\right\}^8.$$

When we take $m = 1$, $\lambda_3 = \frac{1}{2}$, by the transformation $z = (1 + \theta\,\mu)^{\frac{1}{2}}$, (45) reduces to

$$\overline{T}^{(1)}(z) = \frac{2\,(\gamma+1)}{k\,\overline{\theta}}\,(1+z)\,(\Gamma\,z^2)^{-\lambda_2}\times$$

$$\times\Big[\overline{A}\,z - \frac{A_3}{3}\,(2+z)\,\frac{(1-z)}{(1+z)} + (1-z^2)^{\frac{1}{2}}\Big\{\Omega + \frac{A_4}{(b^2-1)^{\frac{1}{2}}}\times$$

$$\times\tan^{-1}\frac{b\,z-1}{\{(b^2-1)\,(1-z^2)\}^{\frac{1}{2}}} + \frac{A_5}{(b^2-1)^{\frac{1}{2}}}\tan^{-1}\frac{1+b\,z}{\{(b^2-1)\,(1-z^2)\}^{\frac{1}{2}}}\Big\}\Big] \quad (46)$$

where $(1+\theta)^{\frac{1}{2}} \leqslant z \leqslant 1$,

$L = C_1 - k\,A_{11}$,

$M = D_1 + E_1 + A_{11}\,(1+k) - k\,B_{11}$,

$N = (1+k)\,B_{11}$,

$$b^2 = \frac{(1+k)\,\theta + k}{k},$$

$$A_1' = \frac{L+M+N}{4\,(b^2-1)},$$

$$A_2 = -\frac{1}{b^2}\Big[(A_1' + A_3)\, b^2 + (A_4 + A_5)\, b - N_1\Big],$$

$$A_3 = \frac{L+M+N}{2\,(b^2-1)},$$

$$A_4 = \frac{L\,b^4 + M\,b^2 + N}{2\,b\,(1-b)\,(1+b)^2}$$

$$A_5 = \frac{L\,b^4 + M\,b^2 + N}{2\,b\,(1+b)\,(1-b)^2},$$

$$\bar{A} = A_1' + A_2,$$

$$B' = A_1' - A_2.$$

$$\Omega = A_2 \left\{ \frac{1-(1+\theta)^{\frac{1}{2}}}{1+(1+\theta)^{\frac{1}{2}}} \right\}^{\frac{1}{2}} + \frac{A_3}{3}\, \frac{\{2+(1+\theta)^{\frac{1}{2}}\}}{\{1+(1+\theta)^{\frac{1}{2}}\}} \times$$

$$\times \left\{ \frac{1-(1+\theta)^{\frac{1}{2}}}{1+(1+\theta)^{\frac{1}{2}}} \right\}^{\frac{1}{2}} - A_1' \left\{ \frac{1+(1+\theta)^{\frac{1}{2}}}{1-(1+\theta)^{\frac{1}{2}}} \right\}^{\frac{1}{2}} -$$

$$- \frac{A_4}{(b^2-1)^{\frac{1}{2}}} \tan^{-1} \frac{b\,(1+\sqrt{\theta})-1}{\{\theta\,(1-b^2)\}^{\frac{1}{2}}} - \frac{A_5}{(b^2-1)^{\frac{1}{2}}} \tan^{-1} \frac{1+b\,(1+\theta)^{\frac{1}{2}}}{\{\theta\,(1-b^2)\}^{\frac{1}{2}}} + B',$$

$$\bar{\theta} = |\,\theta\,|$$

RESULTS AND DISCUSSION

The zeroth order temperature distribution behind the shock, for the plane, cylindrical and spherical pistons for $\alpha = 1$, $\beta = 7$, 8 or 9 is given by (22). For the radiation parameter $\Gamma > 1$, the expression

$$\Big[(1-\Gamma)\,\mu^{1/(j-\omega)\,(n+1)} + \Gamma\Big]$$

increases as μ varies from 1 to 0. $\bar{T}^{(0)}$ decreases as we proceed from the shock to piston. This means that for larger radiation effects the temperature falls off more rapidly towards the piston. When $0 < \Gamma < 1$, the temperature increases towards the piston. When the gas is non-radiating, $\Gamma = 0$ and

$$\bar{T}^{(0)} = \mu^{-1/(j-\omega)\,(n+1)\,(\beta-1)}.$$

Since $\beta > 1$, $j > \omega$, $n > -1$, $\bar{T}^{(0)}$ tends to infinity as $\mu \to 0$. In this case the maximum temperature region is near the piston. When $\Gamma = 1$, $\bar{T}^{(0)} = 1$, that is, the temperature is constant behind the shock wave.

Numerical results show that $T^{(1)}$ contributes significantly for smaller values of Γ and the contribution is more near the piston. When Γ is small, as also pointed earlier by

Wang, the solution does not give good results. We have shown temperature distribution behind the shock in Fig. 1 and 2 for the plane and spherical pistons. For $\Gamma >> 1$, the temperature monotonically increases from the piston to the shock.

When the temperature at the piston is non-zero, (42) shows that the zeroth order solution is the same as for the cool piston but the first order solution is different. The nature of the temperature distribution in this case is the same as when the piston is cool except that now it decreases more rapidly towards the piston. The density is given by (25). The effect of radiation parameter on the density is opposite to that on the temperature. That is, for large values of radiation parameter, the density increases more rapidly towards the piston. The numerical results, obtained by Helliwell[9], show the same trend. We remark that the numerical results obtained by Helliwell[9] for the spherical piston problem with radiation show that in certain cases the optically thin approximation gives nearly the same results as obtained for the case of general opacity, for example, when $\alpha = 1$ and $\beta = 5$, (Fig. 3 of Helliwell[9]). As Γ increases the integrand of (33) decreases and thus the effect of larger radiation is to decrease the distance between the shock and piston.

OPTICALLY THICK MEDIUM

Now we consider the case when the medium is optically thick. Besides (1) and (2) the equations of continuity and momentum, the energy equation is given by

$$\frac{\partial e}{\partial t} + p \frac{\partial \rho^{-1}}{\partial t} = x^{j-1} \frac{\partial \vec{F}^R}{\partial \eta} + \frac{(j-1)}{\rho} \frac{\vec{F}^R}{x} \tag{47}$$

where $\vec{F}^R$ is the net radiative flux. Other basic equations are

$$B = \frac{\sigma}{\pi} T^4 , \tag{48}$$

$$\vec{F}^R = \frac{4\pi}{3} \frac{dB}{d\tau} , \tag{49}$$

$$K = k_1 \rho^\alpha T^\beta , \tag{50}$$

where B is Planck's Function of black body radiation, α, β and k_1 are constants.

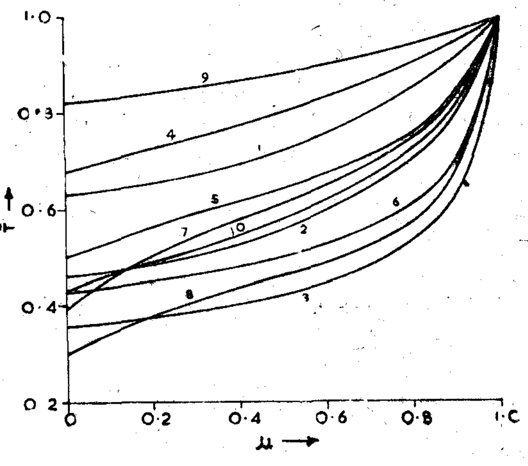

Fig. 1—Temperature distribution behind the shock wave, thin case, $j=3$

1. $\omega = 37/15$, $\Gamma = 5{\cdot}08219$
2. $\omega = 37/15$, $\Gamma = 50{\cdot}8219$
3. $\omega = 37/15$, $\Gamma = 508{\cdot}219$
4. $\omega = 29/15$, $\Gamma = 5{\cdot}08219$
5. $\omega = 29/15$, $\Gamma = 50{\cdot}8$
6. $\omega = 29/15$, $\Gamma = 508$
7. $C' \neq 0$, $\omega = 29/15$, $\Gamma = 50{\cdot}8$
8. $C' \neq 0$, $\omega = 29/15$, $\Gamma = 508$
9. $\omega = 0.333$, $\Gamma = 5{\cdot}24$
10. $\omega = 0{\cdot}333$, $\Gamma = 524{\cdot}27$

From (1), (5), (48) to (50), we obtain

$$\vec{F}^R = \frac{16\,\sigma}{3\,k_1}\,\rho^{1-\alpha}\,T^{3-\beta}\,x^{j-1}\,\frac{\partial T}{\partial \eta} \tag{51}$$

After substituting the value of $\vec{F}^R$ in (47) and using (4), we obtain for $\alpha = 1$

$$c_v\,\frac{\partial T}{\partial t} - \frac{p}{\rho^2}\,\frac{\partial \rho}{\partial t} = \frac{16\,\sigma}{3\,k_1}\left[x^{2(j-1)}\,T^{3-\beta}\,\frac{\partial^2 T}{\partial \eta^2} + (3-\beta)\,x^{2(j-1)}\,T^{2-\beta}\,\times\right.$$
$$\left.\times\left(\frac{\partial T}{\partial \eta}\right)^2 + \frac{2\,(j-1)}{\rho}\,x^{j-2}\,T^{3-\beta}\,\frac{\partial T}{\partial \eta}\right] \tag{52}$$

Substituting the expansions (6) to (9) in (52), we obtain

$$c_v\,\frac{\partial T^{(0)}}{\partial t} = \frac{16\,\sigma}{3\,k_1}\,x^{(0)2(j-1)}\,T^{(0)3-\beta}\left[\frac{\partial^2 T^{(0)}}{\partial \eta^2} + \frac{(3-\beta)}{T^{(0)}}\left(\frac{\partial T^{(0)}}{\partial \eta}\right)^2\right] \tag{53}$$

By the transformation $\xi = t^{m+1}$, (53) reduces to the general form of diffusion equation

$$\frac{\partial T^{(0)}}{\partial \xi} = \theta_2\left[\frac{\partial}{\partial \eta}\left(T^{(0)l}\,\frac{\partial T^{(0)}}{\partial \eta}\right)\right] \tag{54}$$

where $\quad l = 3 - \beta\,,\; m = 2\,(j-1)\,(n+1),$

$$\theta_2 = \frac{16\,\sigma\,c_s^{2(j-1)}}{3\,k_1\,c_v\,(m+1)}$$

We assume a similarity form for the temperature

$$T^{(0)} = L\,\xi^{\bar{\lambda}}\,f(\mu) \tag{55}$$

where $\quad L = \dfrac{2\,\epsilon\,c_s^2\,(n+1)^2}{R\,(\gamma+1)}$

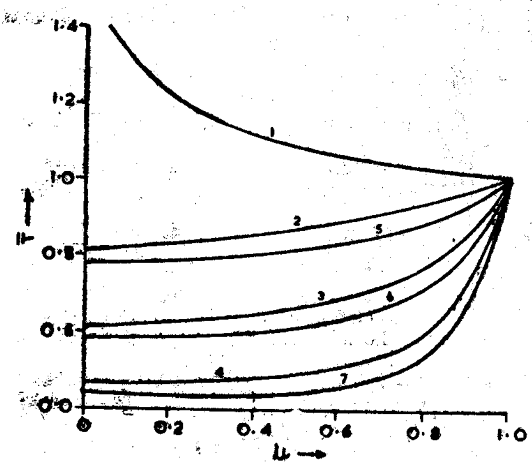

Fig. 2—Temperature distribution behind the shock wave, thin case, $j=1$

1. $\omega = 0$, $\Gamma = 0$
2. $\omega = 0$, $\Gamma = 5.08219$
3. $\omega = 0$, $\Gamma = 50.8219$
4. $\omega = 0$, $\Gamma = 508{\cdot}219$
5. $\omega = 7/15$, $\Gamma = 5.08219$
6. $\omega = 7/15$, $\Gamma = 50{\cdot}8219$
7. $\omega = 7/15$, $\Gamma = 508.219$

and $\bar{\lambda}$ is some constant. The boundary conditions at the piston and the shock in terms of the function f are

$$\left.\begin{array}{l} f(0) = \delta \\ f(1) = 1 \end{array}\right\} \tag{56}$$

The boundary conditions (56) require

$$\bar{\lambda} = \frac{2n}{m+1} \tag{57}$$

The similarity condition for the optically thick medium is

$$\omega = \frac{n(2\beta - 4) + 1}{2(n+1)} \tag{58}$$

Under this condition (54) reduces to

$$f^l \frac{d^2 f}{d\mu^2} + l f^{l-1} \left(\frac{df}{d\mu} \right)^2 + a\mu \frac{df}{d\mu} + b f = 0 \tag{59}$$

where $a = (j - \omega)(n+1)\theta, \; b = -2n\theta$

$$\theta = \frac{3n(n+1)^{2(\beta-3)}}{8(\gamma-1)(j-\omega)^2} \left(\frac{2\epsilon}{\gamma+1} \right)^{\beta-3} \frac{k_1 A^2 R^{4-\beta}}{\sigma c_s^{\left(\frac{5-2\beta}{n+1}\right)}} \tag{60}$$

In an atmosphere with temperature varying from 20,000° to 2,00,000° K, the value of β is 3 and (59) reduces to

$$\frac{d^2 f}{d\mu^2} + a\mu \frac{df}{d\mu} + b f = 0 \tag{61}$$

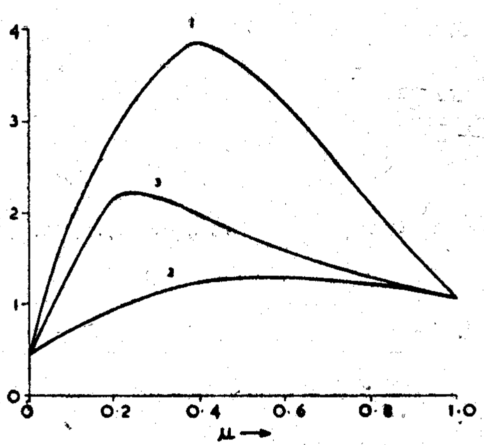

Fig. 3—Temperature distribution behind the shock wave, thick case, $j=1$.

1. $\omega=0$, $\theta=10$
2. $\omega=0{\cdot}4$, $\theta=10$
3. $\omega=0{\cdot}4$, $\theta=50$

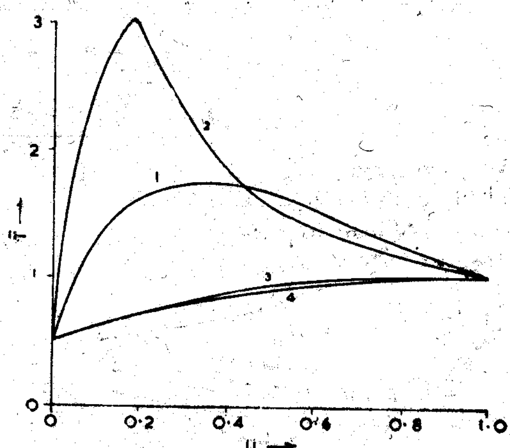

Fig. 4—Temperature distribution behind the shock wave, thick case, $j=3$.

1. $\omega=0$, $\theta=10$
2. $\omega=0$, $\theta=50$
3. $\omega=0{\cdot}6$, $\theta=10$
4. $\omega=0{\cdot}6$, $\theta=50$

We assume $f(\mu) = u(z)$ where $z = -a\mu^2$ and then from (61) we obtain,

$$z \frac{d^2u}{dz^2} + (S - z) \frac{du}{dz} - V z = 0 \tag{62}$$

where $$S = \frac{1}{2} \text{ and } V = \frac{b}{2a}$$

The boundary conditions for $u(z)$ are

$$\left.\begin{aligned} u(z_s) &= 1 \\ u(0) &= \delta \end{aligned}\right\} \tag{63}$$

The solution of (63) is obtained in terms of confluent hypergeometric function as

$$u(z) = c_1\, {}_1F_1 [V, S, z] + c_2 z^{1-S}\, {}_1F_1 [1 + V - S, 2 - S, z] \tag{64}$$

where c_1 and are c_2 are arbitrary constants and

$${}_1F_1 \Big[V, S, z\Big] = \sum_0^\infty A_k z^k, \tag{65}$$

$$A_k = \frac{V (V+1) \ldots\ldots\ldots (V + k - 1)}{S(S+1) \ldots\ldots\ldots (S+k-1)\, k!}. \tag{66}$$

Using the boundary conditions (63) we obtain,

$$u(z) = \delta\, {}_1F_1 [V, S, z] + \frac{\{1 - \delta\, {}_1F_1 [V, S, z_s]\}}{z_s^{1-S}\, {}_1F_1 [1 + V - S, 2 - S, z_s]} \times \tag{67}$$
$$\times z^{1-S}\, {}_1F_1 [1 + V - S, 2 - S, z]$$

If we take cool piston so that $\delta = 0$, the temperature $T^{(0)}$ is given by

$$\frac{T^{(0)}}{L\xi^\lambda} = u(z) = \left(\frac{z}{z_s}\right)^{1-S} \frac{{}_1F_1 [1+V-S, 2-S, z]}{{}_1F_1 [1+V-S, 2-S, z_s]} \tag{68}$$

RESULTS AND DISCUSSION

We have given numerical results in Fig. 3 and 4 for plane and spherical pistons respectively both for the homogeneous and inhomogeneous media. We have used Slater's Table[12] for confluent hypergeometric function for smaller values of z and for $z > 10$ we have used the asymptotic expansion

$${}_1F_1 [V, S\, z] \approx \frac{(S-1)!}{(S-V-1)!} (-z)^V \left\{1 - \frac{V(V-S)}{z} + \right.$$
$$\left. + \frac{V(V+1)(V-S+1)(V-S+2)}{2!\, z^2} + \ldots \right\} +$$
$$+ \frac{(S-1)!}{(V-1)!} e^z z^{V-S} \left\{1 + \frac{(1-V)(S-V)}{z} + \ldots\right\} \tag{69}$$

Since in our case z is always negative, we use the transformation

$${}_1F_1 (V, S, -z) = e^{-z}\, {}_1F_1 [S - V, S, z] \tag{70}$$

Our numerical results show that in case of plane piston the temperature behind the shock rises to a maximum and then falls off towards the piston. In case of spherical piston, when the medium is homogeneous, the nature of temperature distribution is the same as in the plane piston but when the medium is inhomogeneous the temperature behind the shock falls monotonically towards the piston. Since $S=\frac{1}{2}$ there is no singularity at the piston $z=0$ in the solution (56). As in the thin case the distance between shock and piston decreases due to larger radiation effect. When the medium ahead of the shock is homogeneous the effect of increase of opacity behind the shock is to reduce the temperature hump behind the shock so that the flow behind the shock becomes isothermal for large opacity[2].

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REFERENCES

1. SOBOLEV, V. V., "A Treatise on Radiative Transfer", (Von Nostrand Company, New York), 1963.
2. ELLIOT, L. A., *Proc. Roy. Soc.*, **A 258** (1960), 287.
3. WANG, K. C., *J. Fluid Mech.*, **20** (1964), 447.
4. WANG, K. C., "One Dimensional Shock Propagation with Thermal Radiation", Research Report, RR-47, Martin, Baltimore, March 1965.
5. TRAUGOTT, S. C., "A Differential Approximation for Radiative Transfer with Application to Normal Shock Structure", RR-34, December 1962.
6. WANG, K. C., *Physics of Fluids*, **9** (1966), 1922.
7. MARSHAK, R. E., *Physics of Fluids*, **1** (1958), 24.
8. CHERNYI, G. G., "Introduction to Hypersonic Flow", (Academic Press Inc., New York) (1961), 146.
9. HELLIWELL, J. B., *J. Fluid Mech.*, **37** (1969), 497.
10. BLOOR, M. I. G., *J. Fluid Mech.*, **29** (1967), 485.
11. SACHDEV, P. L. & ASHRAF, S., *Indian J. Pure Appl. Maths.*, **1** (1970), 579.
12. SLATER, L. J., "Confluent Hypergeometric Functions", (University Press, Cambridge), 1960.