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Transient Heat Transfer in a Slab with Heat Generation

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Abstract. A numerical solution has been obtained for transient thermal distribution in a slab in which chemical, electrical or nuclear energy is converted into thermal energy. An *implicit* scheme is used to set up the finite difference analog of the diffusion equation and the accompanying initial and boundary conditions.

Nomenclature

a	Thermal diffusivity
b	$= 2(\Delta X)^2/\Delta\tau$
A	Heat generation per unit time per unit volume
K	thermal conductivity
l	length of the slab
n	number of nodal points
p	Heat source parameter $(Al^2/2w_0K)$
T	Temperature within the slab
t	time
w_0	temperature at $x = l$
W	Nondimensional temperature
$W_{i,m}$	Nondimensional temperature (w/w_0) at ith nodal point and mth time step.
x	coordinate along the length of the slab
X	Nondimensional length (x/l)
τ	Nondimensional time (Fourier number)

1. Introduction

Transient thermal distribution studies are very important in atmospheric, earth, biological, geo-fluid-dynamic and technological sciences. Structural technology relies heavily on thermal diffusion studies in material selection for re-entry shields, chemical and thermal reactor components, combustion devices, gun barrels etc. Because of its great applicability, it has drawn the attention of many researchers[1-4].

This investigation deals with the study of thermal diffusion in a slab with heat generation, which models the heating of a slab by the passage of a current. Carslaw and Jaeger[5] have carried out this study but the solution is in terms of a series expansion which hold for small time only. An attempt has been made to trace the evolution of the steady state temperature distribution. The response of the evolution of the temperature distribution to the variations of heat sources and material properties has also been worked out.

The diffusion equation with the forcing term provided by the heat generation and the accompanying initial and boundary conditions are expressed as a set of finite difference equations through Crank & Nicholson[6] implicit scheme which in turn are solved by using Thomas[7] algorithm. The computer algorithm developed for the solution of this problem can cater for different number of nodal points. The size of the time interval is stepped up as the solution approaches towards steady state. As the finite difference equations are set up with the help of Crank-Nicholson implicit scheme, which is second order correct, a small number of nodal points gives results that are correct upto third decimal place.

2. Formulation of the Problem

Let a slab extending from $x = 0$ to $x = l$ be maintained at temperatures 0 and w_0 at both the ends with constant rate of heat generation within it because of dielectric heating. If the heat is supposed to be produced at a constant rate A per unit time per unit volume, the energy equation modelling the physical system becomes

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{a}\frac{\partial T}{\partial t} = -\frac{A}{K} \tag{1}$$

where T, t, a, A and K are temperature, time, diffusivity, thermal heat generation per unit time per unit volume and thermal conductivity respectively.

The initial and boundary conditions are

$$T = 0 \text{ for all } x \text{ for } t = 0 \tag{2}$$

$$T = 0 \text{ at } x = 0 \text{ for } t = > 0 \tag{3}$$

$$T = w_0 \text{ at } x = l \text{ for } t > 0 \tag{4}$$

Making use of the transformation

$$T = \frac{A(l^2 - x^2)}{2K} + w \tag{5}$$

Eqn. (1) and the associated initial and boundary conditions transform to the following equations :

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{a}\frac{\partial w}{\partial t} = 0 \tag{6}$$

$$w = -\frac{A(l^2 - x^2)}{2K} \text{ for all } x \text{ and } t = 0 \tag{7}$$

$$w = -\frac{Al^2}{2K} \text{ for } x = 0 \text{ and } t > 0 \tag{8}$$

$$w = w_0 \text{ for } x = l \text{ and } t > 0 \tag{9}$$

Equations (6)–(9) through the transformations

$$X = \frac{x}{l}, \; W = \frac{w}{w_0} \text{ and } \tau = \frac{at}{l^2} \tag{10}$$

transform to the nondimensional form

$$\frac{\partial^2 W}{\partial X^2} = \frac{\partial W}{\partial \tau} \tag{11}$$

$$W = -p(1 - X^2) \text{ for all } X \text{ and } \tau = 0 \tag{12}$$

where

$$p = \frac{Al^2}{2w_0K}$$

$$W = -\frac{1}{2}p \text{ for } X = 0 \text{ and } \tau > 0 \tag{13}$$

$$W = 1 \text{ for } X = 1 \text{ and } \tau > 0 \qquad ((14)$$

The finite difference analogue of the energy equation (11) and the accompanying initial and boundary conditions (12)–(14), solution of the set of the difference equations and discussion of the numerical results are contained in the subsequent sections.

3. Solution of the Problem

The finite difference equations are set up by spacing the nodal points in such a way that the value of the independent variable at different nodal points is given by

$$X_i = i \, \Delta X$$

where $\Delta X = \frac{1}{n}$, n being the number of intervals in which the region 0 to 1 is divided.

Using Crank-Nicholson's implicit scheme and denoting the nondimensional temperature at the ith nodal point and mth time step by $W_{i,m}$, the finite difference analogue of Eqn. (11) is

$$W_{i-1,m+1} - (2+b)\,W_{i,m+1} + W_{i+1,m+1} = -W_{i-1,m} + (2-b)\,W_{i,m} - W_{i+1,m} \tag{15}$$

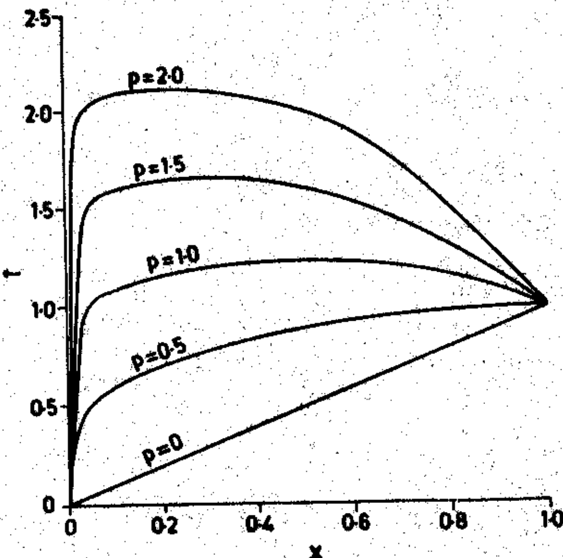

Figure 1. Steady state temperature for different rates of heat generation.

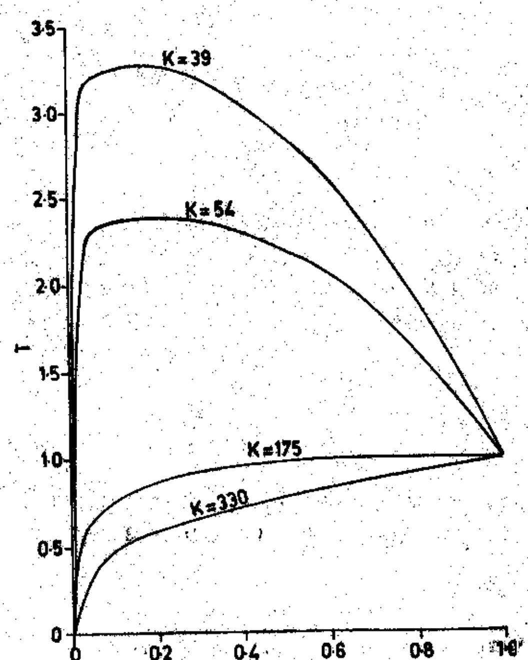

Figure 2. The effect of thermal conductivity on the temperature distribution.

where

$$b = \frac{2(\Delta x)^2}{\Delta \tau} \text{ for } 2 \leqslant i \leqslant n - 2$$

This gives a set of $n - 3$ equations for $n - 1$ nodal points. The set of equations can, however, be made complete by using the boundary conditions (13) and (14) which give the following finite difference equations :

$$-(2 + b) W_{1,m+1} + W_{2,m+1} = (2 - b) W_{1,m} - W_{2,m} \tag{16}$$

$$W_{n-2,m+1} - (2 + b) W_{n-1,m+1} = - W_{n-2,m} + (2 - b) W_{n-1,m} - 2 \tag{17}$$

It may be observed that the right hand sides of Eqns. (15)–(17) are known through Eqn. (12). Thus these $n - 1$ linear algebraic equations form a tri-diagonal system, which can be conveniently analysed by Thomas' algorithm. The computer algorithm that has been developed can cater for different step sizes. The size of the time interval has been stepped up as the solution marches towards steady state. In the

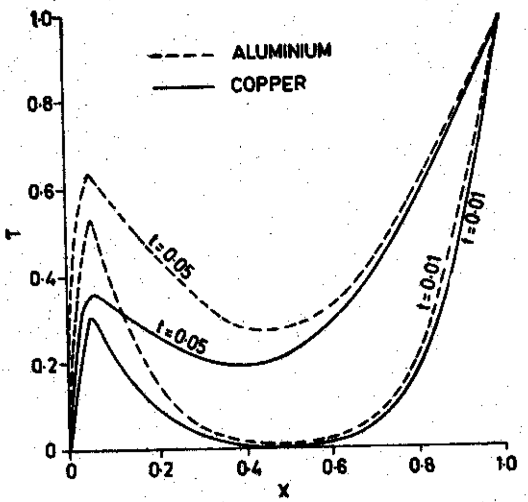

Figure 3. Comparison of temperature distribution in aluminium and copper for different times.

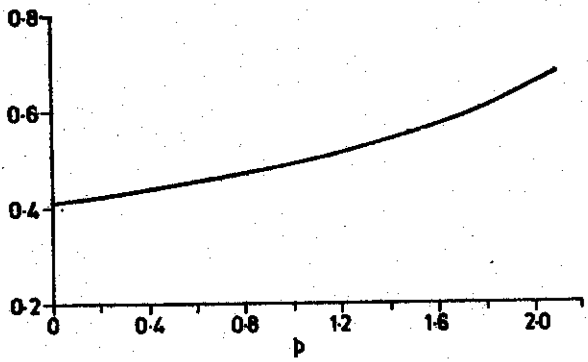

Figure 4. Time required to reach steady state *vs* rate of heat generation.

present computation the value of n is taken as 20 which gives results correct upto third decimal place.

Numerical results and response of the thermal distribution to various physical parameters are discussed in the section (4).

A few representative curves have been depicted but the computation for complete spectrum of values of physical parameters can be carried out using the algorithm developed.

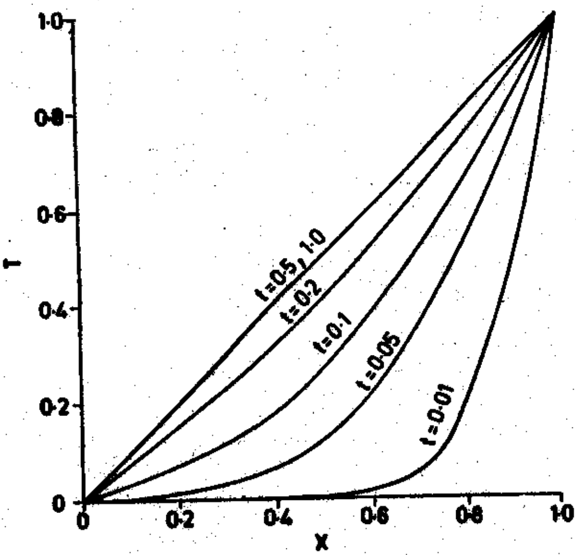

Figure 5. Time history of temperature in the slab without heat sources.

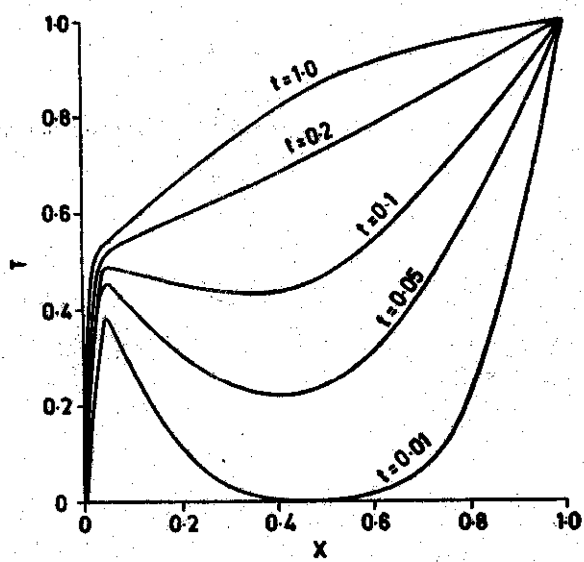

Figure 6. Time history of temperature with heat source $p = 0.5$.

4. Results and Discussion

Figure 1 depicts that the steady state temperature increases with the increase in the rate heat of generation. The temperature profile is linear when there is no heat generation, which agrees with the steady state solution of the energy equation without heat sources and this adds to the confidence in the computations carried out in this paper.

The steady state temperature distribution in the slabs of different thermal conductivities is shown in Fig 2. Increase in the temperature levels with the decrease in the thermal conductivity is in keeping with the physical behaviour exhibited by Fig 1, because decrease in the value of the thermal conductivity results in increase in the value of p.

The temperature distribution in aluminium and copper slabs for two typical instants of time when the heat generation is specified by $p = 0.5$. is depicted in Fig. 3.

Fig. 4 exhibits that the time required for the evolution of steady state temperature distribution in a slab increases with the increase in the rate of heat generation. The detailed evolution of temperature distribution with and without heat sources respectively is explained in Figs. 5 and 6. These figures show that the time required to achieve steady state is higher when heat is generated within it which is also supported by Fig. 4.

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