FOURIER SERIES FOR KAMPÉ DE FÉRIET FUNCTION

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Some integrals involving Kampé de Fériet function have been evaluated. These integrals have further been employed to obtain some Fourier series for Kampé de Fériet functions. Some particular cases have also been discussed.

For the sake of brevity and simplicity Kampé de Fériet function[1] is represented as

$$F\begin{matrix} g:h;H \\ \\ i:j;J \end{matrix}\left[\begin{matrix} a_g : b_h ; B_H ; \\ & x, y \\ \alpha_i : c_j ; C_J ; \end{matrix}\right] = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\Pi (a_g)_{p+q} \, \Pi (b_h)_p \, \Pi (B_H)_q \, x^p \, y^q}{\Pi (\alpha_i)_{p+q} \, \Pi (c_j)_p \, \Pi \, (C_J)_q \, p! \, q!} \tag{1}$$

where μ_i denotes the set of parameters $\mu_1, \ldots, \mu_i$. $\Pi (\mu_i)_s$ denotes the product $(\mu_1)_s \ldots \ldots (\mu_i)_s$, $(\mu)_s$ denotes $\overline{|(\mu+s)}/\overline{|s}$. Colon (:) and semicolon (;) separate the terms of the form $(\alpha_i)_{p+q}$ and $(C_J) \ldots (c_j)_r$, $(C_J)_q$ etc. In what follows the symbol $\triangle(\delta, \alpha)$ represents the set of parameters $\alpha/\delta, \ldots, (\alpha+1)/\delta, \ldots, (\alpha+\delta-1)/\delta$ where δ is a positive integer.

Ragab[2] has shown that the series on the right of (1) converges for all finite x, y if $g+h \leqslant i+j$, $g+H \leqslant i+J$, converges for $|x|+|y| < \min(1, 2^{i-g+1})$ if $g+h = i+j+1$, $g+H = i+J+1$. It can easily be shown that the series converges if

$$g+h = i+j+1, \quad g+H = i+J+1$$

and $$|x| \leqslant 1, \quad |y| < 1, \quad Re(X) > 0,$$

or $$|x| < 1, \quad |y| \leqslant 1, \quad Re(Y) > 0,$$

or $$|x| \leqslant 1, \quad |y| \leqslant 1, \quad Re(X) > 0, Re(Y) > 0.$$

where

$$X = \Sigma\alpha_i + \Sigma c_i - \Sigma a_g - \Sigma b_h,$$

$$Y = \Sigma\alpha_i + \Sigma C_J - \Sigma a_g - \Sigma B_H,$$

Following formulae will be required in the proofs:

(*a*) If δ is a positive integer then

$$\Gamma(\alpha+\delta n)=\Gamma(\alpha)\,\delta^{\delta n}\prod_{i=0}^{\delta-1}\left(\frac{\alpha+i}{\delta}\right)_n \tag{2}$$

(*b*) The integrals[3,4]

$$\int_0^{\pi}(\sin t)^{\rho}\cos ut\,dt=\frac{\pi\,\Gamma(1+\rho)\cos\frac{\pi u}{2}}{2^{\rho}\Gamma\left(1+\frac{\rho+u}{2}\right)\Gamma\left(1+\frac{\rho-u}{2}\right)}=A(u)\ (\text{say}); \tag{3}$$

$$\int_0^{\pi}(\sin t)^{\rho}\sin ut\,dt=\frac{\pi\,\Gamma(1+\rho)\sin\frac{\pi u}{2}}{2^{\rho}\Gamma\left(1+\frac{\rho+u}{2}\right)\Gamma\left(1+\frac{\rho-u}{2}\right)}=B(u)\ (\text{say}); \tag{4}$$

$$\int_0^{\pi}(\cos t)^{\rho}\cos ut\,dt=\frac{\pi\,\Gamma(1+\rho)}{2^{\rho}\Gamma\left(1+\frac{\rho+u}{2}\right)\Gamma\left(1+\frac{\rho-u}{2}\right)}=C(u)\ (\text{say}); \tag{5}$$

$Re(\rho)>0$.

THE INTEGRALS

The integrals to be established are:

$$\int_0^{\pi}(\sin t)^{\rho}\cos ut\,F\begin{matrix} g:h;H \\ i:j;J\end{matrix}\left[\begin{matrix} a_g:b_h;B_H; \\ \alpha_i:c_j;C_J;\end{matrix}\ x(\sin t)^{2\delta},\,y(\sin t)^{2\delta}\right]dt=A(u)F(u) \tag{6}$$

$$\int_0^{\pi}(\sin t)^{\rho}\cos ut\,F\begin{matrix} g:h;H \\ i:j;J\end{matrix}\left[\begin{matrix} a_g:b_h;B_H; \\ \alpha_i:c_j;C_J;\end{matrix}\ x(\sin t)^{2\delta},\,y\right]dt=A(u)F_1(u) \tag{7}$$

$$\int_0^{\pi}(\sin t)^{\rho}\cos ut\,F\begin{matrix} g:h;H \\ i:j;J\end{matrix}\left[\begin{matrix} a_g:b_h;B_H; \\ \alpha_i:c_j;C_J;\end{matrix}\ x,\,y(\sin t)^{2\delta}\right]dt=A(u)F_2(u) \tag{8}$$

$$\int_0^{\pi} (\sin t)^{\rho} \sin ut \, F \begin{matrix} g:h;H \\ i:j;J \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix} \; x (\sin t)^{2\delta}, \, y (\sin t)^{2\delta} \right] dt = B(u) \, F(u) \quad (9)$$

$$\int_0^{\pi} (\sin t)^{\rho} \sin ut \, F \begin{matrix} g:h;H \\ i:j;J \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix} \; x (\sin t)^{2\delta}, \, y \right] dt = B(u) F_1(u) \quad (10)$$

$$\int_0^{\pi} (\sin t)^{\rho} \sin ut \, F \begin{matrix} g:h;H \\ i:j;J \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix} \; x, \, y (\sin t)^{2\delta} \right] dt = B(u) F_2(u) \quad (11)$$

$$\int_0^{\pi} (\cos t)^{\rho} \cos ut \, F \begin{matrix} g:h;H \\ i:j;J \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix} \; x (\cos t)^{2\delta}, \, y (\cos t)^{2\delta} \right] dt = C(u) \, F(u) \quad (12)$$

$$\int_0^{\pi} (\cos t)^{\rho} \cos ut \, F \begin{matrix} g:h;H \\ i:j;J \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix} \; x (\cos t)^{2\delta}, \, y \right] dt = C(u) F_1(u) \quad (13)$$

$$\int_0^{\pi} (\cos t)^{\rho} \cos ut \, F \begin{matrix} g:h;H \\ i:j;J \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix} \; x, \, y (\cos t)^{2\delta} \right] dt = C(u) F_2(u) \quad (14)$$

where the parameters etc. of the Kampé de Fériet functions are so restricted that they converge. $A(u)$, $B(u)$ and $C(u)$ have respectively been defined in (3) to (5) and

$$\left.\begin{aligned}
F(u) &= F \begin{matrix} g+2\delta : h ; H \\ i+2\delta : j ; J \end{matrix} \left[\begin{matrix} a_g, \triangle(2\delta, 1+\rho) : b_h ; B_H ; \\ \alpha_i, \triangle\left(\delta, 1+\frac{\rho+u}{2}\right), \triangle\left(\delta, 1+\frac{\rho-u}{2}\right); c_j ; C_J ; \end{matrix} \; x, y \right] \\
F_1(u) &= F \begin{matrix} g : h+2\delta ; H \\ i : j+2\delta ; J \end{matrix} \left[\begin{matrix} a_g : b_h, \triangle(2\delta, 1+\rho) ; B_H ; \\ \alpha_i : c_j, \triangle\left(\delta, 1+\frac{\rho+u}{2}\right), \triangle\left(\delta, 1+\frac{\rho-u}{2}\right); C_J ; \end{matrix} \; x, y \right] \\
F_2(u) &= F \begin{matrix} g : h ; H+2\delta \\ i : j ; J+2\delta \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H, \triangle(2\delta, 1+\rho) ; \\ \alpha_i : c_j ; C_J, \triangle\left(\delta, 1+\frac{\rho+u}{2}\right), \triangle\left(\delta, 1+\frac{\rho-u}{2}\right); \end{matrix} \; x, y \right]
\end{aligned}\right\} \quad (15)$$

Proof:—To establish (6), expressing the Kampé de Fériet function in the integrand as in (1) and interchanging the order of summation and integration which is justified[5], we have

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\Pi (a_g)_{p+q} \, \Pi (b_h)_p \, \Pi (B_H)_q \, x^p \, y^q}{p \, ! \, q \, ! \, \Pi (\alpha_i)_{p+q} \, \Pi (c_j)_p \, \Pi (C_J)_q} \int_0^{\pi} (\sin t)^{\rho + 2\delta p + 2\delta q} \cos ut \, dt$$

Now evaluating the integral with the help of (3) using (1) and (2), we get the right hand side of (6). Formulae (7) and (8) can similarly be proved. The sets of formulae (9) to (14) can be established by using the same procedure with the help of (4) and (5).

Particular Cases

Treating a_g and α_i as empty parameters, it is seen that the Kampé de Fériet functions of the integrals reduce to the products of the pairs of hypergeometric functions. This enables to have the values of the integrals involving the products of hypergeometric functions. To illustrate, taking (12), which in the absence of parameters a_g, α_i reduces to the form

$$\int_0^{\pi} (\cos t)^{\rho} \cos ut \; {}_hF_j \left(b_h \,;\, c_j \,;\, x (\cos t)^{2\delta}\right) \; {}_HF_J \left(B_H \,;\, C_J \,;\, y (\cos t)^{2\delta}\right) dt$$

$$= C(u) \, F \begin{matrix} 2\delta : h \,;\, H \\ 2\delta : j \,;\, J \end{matrix} \left[\begin{matrix} \triangle (2\delta \,,\, 1 + \rho) : b_h \,;\, B_H \\ \triangle \left(\delta \,,\, 1 + \frac{\rho + u}{2}\right), \triangle \left(\delta \,,\, 1 + \frac{\rho - u}{2}\right) : c_j \,;\, C_J \,; \end{matrix} x, y \right] \quad (16)$$

For $\delta = 1$, (16) agrees with the result obtained by Sexana & Vyas[6].

EXPANSION FORMULAE

The expansion formulae to be obtained are

$$(\sin t)^{\rho} \, F \begin{matrix} g : h \,;\, H \\ i : j \,;\, J \end{matrix} \left[\begin{matrix} a_g : b_h \,;\, B_H \,; \\ \alpha_i : c_j \,;\, C_J \,; \end{matrix} \; x (\sin t)^{2\delta}, \;\; y (\sin t)^{2\delta} \right]$$

$$= \frac{1}{\pi} A(0) \, F(0) + \frac{2}{\pi} \sum_{r=1}^{\infty} A(r) \, F(r) \cos rt \quad (17)$$

$$= \frac{2}{\pi} \sum_{r=0}^{\infty} B(r) \, F(r) \sin rt \quad (18)$$

$$(\sin t)^{\rho}\ F\begin{matrix} g:h;H \\ \\ i:j;J \end{matrix}\left[\begin{matrix} a_g : b_h ; B_H ; \\ \\ \alpha_i : c_j ; C_J ; \end{matrix}\ x\,(\sin t)^{2\delta}, y\right]$$

$$= \frac{1}{\pi} A(0)\, F_1(0) + \frac{2}{\pi} \sum_{r=1}^{\infty} A(r)\, F_1(r) \cos rt \tag{19}$$

$$= \frac{2}{\pi} \sum_{r=0}^{\infty} B(r)\, F_1(r) \sin rt \tag{20}$$

$$(\sin t)^{\rho}\ F\begin{matrix} g:h;H \\ \\ i:j;J \end{matrix}\left[\begin{matrix} a_g : b_h ; B_H ; \\ \\ \alpha_i : c_j ; C_J ; \end{matrix}\ x, y\,(\sin t)^{2\delta}\right]$$

$$= \frac{1}{\pi} A(0)\, F_2(0) + \frac{2}{\pi} \sum_{r=1}^{\infty} A(r)\, F_2(r) \cos rt \tag{21}$$

$$= \frac{2}{\pi} \sum_{r=0}^{\infty} B(r)\, F_2(r) \sin rt \tag{22}$$

$$(\cos t)^{\rho}\ F\begin{matrix} g:h;H \\ \\ i:j;J \end{matrix}\left[\begin{matrix} a_g : b_h ; B_H ; \\ \\ \alpha_i : c_j ; C_J ; \end{matrix}\ x\,(\cos t)^{2\delta}, y\,(\cos t)^{2\delta}\right]$$

$$= \frac{1}{\pi} C(0)\, F(0) + \frac{2}{\pi} \sum_{r=1}^{\infty} C(r)\, F(r) \cos rt, \tag{23}$$

$$(\cos t)^{\rho}\ F\begin{matrix} g:h;H \\ \\ i:j;J \end{matrix}\left[\begin{matrix} a_g : b_h ; B_H ; \\ \\ \alpha_i : c_j ; C_J ; \end{matrix}\ x\,(\cos t)^{2\delta}, y\right]$$

$$= \frac{1}{\pi} C(0)\, F_1(0) + \frac{2}{\pi} \sum_{\gamma=1}^{\infty} C(r)\, F_1(r) \cos rt, \tag{24}$$

$$(\cos t)^{\rho}\, F\begin{matrix} g:h;H \\ i:j;J \end{matrix}\left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix}\; x\,,\; y\,(\cos t)^{2\delta}\right]$$

$$= \frac{1}{\pi}\, C(0)\, F_2(0) + \frac{2}{\pi} \sum_{r=1}^{\infty} C(r)\, F_2(r) \cos rt, \qquad (25)$$

where the parameters etc. of the Kampé de Fériet functions are so restricted that they converge. $A(r)$, $B(r)$ and $C(r)$ are defined in (2) to (4); $F(r)$, $F_1(r)$ and $F_2(r)$ have been defined in (15).

Proof:—

Let
$$f(t) = (\sin t)^{\rho}\, F\begin{matrix} g:h;H \\ i:j;J \end{matrix}\left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix}\; x\,(\sin t)^{2\delta}\,,\; y\,(\sin t)^{2\delta}\right] \qquad (26)$$

$$= \frac{C_0}{2} + \sum_{r=1}^{\infty} C_r \cos rt$$

Equation (26) is valid since $f(t)$ is continuous and of bounded variation in the interval $0 \leqslant t \leqslant \pi$. Multiplying both sides by $\cos ut$ and integrating with respect to t from 0 to π we have

$$\int_0^{\pi} (\sin t)^{\rho} \cos ut\, F\begin{matrix} g:h;H \\ i:j;J \end{matrix}\left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix}\; x\,(\sin t)^{2\delta}\,,\; y\,(\sin t)^{2\delta}\right] dt$$

$$= \frac{C_0}{2} \int_0^{\pi} \cos ut\, dt + \sum_{r=1}^{\infty} C_r \int_0^{\infty} \cos rt \cos ut\, dt \qquad (27)$$

Now using (6) and the orthogonality property of cosine functions, we get

$$C_u = \frac{2}{\pi}\, A(u)\, F(u). \qquad (28)$$

Formula (17) now follows from (26) and (28).

Fourier series (19), (21) and (23) to (25) can similarly be established.

To prove (18), let

$$g(t) = (\sin t)^{\rho} F \begin{matrix} g:h;H \\ i:j;J \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J \end{matrix} \; x (\sin t)^{2\delta}, y (\sin t)^{2\delta} \right] \tag{29}$$

$$= \sum_{r=0}^{\infty} C_r \sin rt$$

Equation (29) is valid since $g(t)$ is continuous and of bounded variation in the interval $0 \leqslant t \leqslant \pi$. Multiplying both sides by *sin ut* and integrating with respect to t from 0 to π, we have

$$\int_0^{\pi} (\sin t)^{\rho} \sin ut \; F \begin{matrix} g:h;H \\ i:j;J \end{matrix} \left[\begin{matrix} a_g : b_h ; B_H ; \\ \alpha_i : c_j ; C_J ; \end{matrix} \; x (\sin t)^{2\delta}, y (\sin t)^{2\delta} \right]$$

$$= \sum_{r=0}^{\infty} C_r \int_0^{\pi} \sin rt \sin ut \, dt$$

Now using the orthogonality property of sine functions and the formula (9), we have

$$C_r = \frac{2}{\pi} B(r) F(r)$$

From (29) and (30), we get (18)

Fourier series (20) and (22) are similarly obtained.

Particular Cases

In view of the remarks in the particular cases discussed earlier it can be concluded that a Fourier series for the product of hypergeometric functions can be obtained as particular cases of the formulae (17) to (25). As an example (23), in the absence of the parameters a_g and α_i, takes the form

$$(\cos t)^{\rho} \, {}_hF_j [\, b_h ; c_j ; x (\cos t)^{2\delta} \,] \, {}_HF_J [\, B_H ; C_J ; y (\cos t)^{2\delta} \,] = \frac{1}{\pi} C(0) F(0) +$$

$$+ \frac{2}{\pi} \sum_{u=0}^{\infty} C(u) F \begin{matrix} 2\delta:h;H \\ 2\delta:j;J \end{matrix} \left[\begin{matrix} \triangle(2\delta, 1+\rho) : b_h ; B_H \\ \triangle\left(\delta, 1+\frac{\rho+u}{2}\right), \triangle\left(\delta, 1+\frac{\rho-u}{2}\right) : c_j ; C_J \end{matrix} \; x, y \right] \tag{30}$$

which agrees with the earlier result[7] for $\delta = 1$.

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