MINIMUM IMPULSE TRAJECTORIES WITH SPECIFIED TRANSFER TIME

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Transfer Trajectories between circular orbits in an inverse square gravitational field are investigated, the criterion of optimization adopted being minimization of total impulse velocity applied at the terminals with flight time of the rocket serving as a constraint.

Lee & Florence[1] have discussed inter-orbital transfer trajectories with specified total impulse velocity which require minimum flight time in the transfer operation. An analysis of the inter-orbital transfer trajectories of specified flight time requiring minimum impulse velocity at the initial and final terminals which corresponds to minimum fuel consumption has been presented.

NOMENCLATURE

e = eccentricity

q = peri-apsis distance, in radii of initial orbit

n = ratio of radius of final orbit to radius of initial orbit

T = transfer time, in initial orbit periodic time

V = Total impulse velocity, in units of initial orbit circular velocity

V_{E1} = surface escape velocity of the gravitating body in the initial orbit, in units of initial orbit circular velocity

V_{E2} = surface escape velocity of the gravitating body in the final orbit, in units of initial orbit circular velocity.

ANALYSIS

If the radius, circular velocity and periodic time are taken corresponding to the initial orbit as the units of distance, velocity and time, the total impulse velocity V required in the transfer operation of the rocket and the transfer time T will be given[1] by

$$V = \left[3 - 2\{q(1+e)\}^{\frac{1}{2}} - \frac{1-e}{q} + V^2_{E1}\right]^{\frac{1}{2}} + \left[\frac{3 - 2\left(\frac{q(1+e)}{n}\right)^{\frac{1}{2}}}{n} - \frac{1-e}{q} + V^2_{E2}\right]^{\frac{1}{2}} \qquad (1)$$

and

$$T = \frac{1}{2\pi}\left[\frac{q}{1-e}\right]^{3/2}\left(\cos^{-1}\left[\frac{q-n(1-e)}{q\,e}\right] - \cos^{-1}\left[\frac{q-(1-e)}{q\,e}\right] + \right.$$

$$\left. + e\left\{\sin\left[\cos^{-1}\left(\frac{q-(1-e)}{q\,e}\right)\right] - \sin\left[\cos^{-1}\left(\frac{q-n(1-e)}{q\,e}\right)\right]\right\}\right) \quad (2)$$

The total impulse velocity is to be minimized subject to the transfer time T by the method of Lagrange. The equations giving the extremized impulse are

$$\frac{\partial T}{\partial e}\cdot\frac{\partial V}{\partial q} - \frac{\partial T}{\partial q}\cdot\frac{\partial V}{\partial e} = 0 \quad (3)$$

and

$$T - T_0 = 0 \quad (4)$$

Substituting from (1) and (2) in (3), which on differentiation and simplification gives

$$\frac{P_1\left(\frac{1-e}{q}\right)^2\left(\frac{1}{Q_2}-\frac{n^2}{Q_1}\right)+\frac{3}{4\pi}\,(P_2+eP_3)\frac{q^{\frac{1}{2}}}{(1-e)^{3/2}}}{\frac{P_1}{e}\left[\frac{1-e}{q}\left(\frac{1-q}{Q_2}-\frac{n(n-q)}{Q_1}\right)+e\,P_3\right]+\frac{3}{4\pi}\,(P_2+eP_3)\frac{q^{3/2}}{(1-e)^{3/2}}}$$

$$= \frac{\frac{1}{\sqrt{R_1}}\left(\frac{1-e}{q^2}-\left\{\frac{1+e}{q}\right\}^{\frac{1}{2}}\right)+\frac{1}{\sqrt{R_2}}\left(\frac{1-e}{q^2}-\frac{1}{n^{3/2}}\left\{\frac{1+e}{q}\right\}^{\frac{1}{2}}\right)}{\frac{1}{\sqrt{R_1}}\left(\frac{1}{q}-\left\{\frac{q}{1+e}\right\}^{\frac{1}{2}}\right)+\frac{1}{\sqrt{R_2}}\left(\frac{1}{q}-\frac{1}{n^{3/2}}\left\{\frac{q}{1+e}\right\}^{\frac{1}{2}}\right)} \quad (5)$$

where

$$P_1 = \frac{1}{2\pi}\left[\frac{q}{1-e}\right]^{3/2}$$

$$P_2 = \cos^{-1}\left[\frac{q-n(1-e)}{q\,e}\right] - \cos^{-1}\left[\frac{q-(1-e)}{q\,e}\right]$$

$$P_3 = \sin\left[\cos^{-1}\left(\frac{q-(1-e)}{q\,e}\right)\right] - \sin\left[\cos^{-1}\left(\frac{q-n\,(1-e)}{q\,e}\right)\right]$$

$$Q_1 = [(q\,e)^2 - \{q-n\,(1-e)\}^2]^{\frac{1}{2}}$$

$$Q_2 = [(q\,e)^2 - \{q-(1-e)\}^2]^{\frac{1}{2}}$$

$$R_1 = \left[3 - 2\,\{q\,(1+e)\}^{\frac{1}{2}} - \frac{1-e}{q}\right] + V^2_{E1}$$

$$R_2 = \left[\frac{3-2\left\{\dfrac{q\,(1+e)}{n}\right\}^{\frac{1}{2}}}{n} - \frac{1-e}{q}\right] + V^2_{E2}$$

Equations (4) and (5) are two transcendental equations in two unknowns e and q which can be solved numerically for a set of given values of n, V_{E1}, V_{E2} and T_0. Having known e and q, (1) will give minimum impulse velocity required in the transfer operation. As a numerical illustration of the above general analysis, (4) and (5) have been solved for Earth-Mars transfer for different values of T_0. The results obtained are shown graphically in Fig. 1 and 2. A study of Fig. 1 shows an interesting result that the rate of variation of eccentricity with respect to peri-apsis distance of the transfer trajectory is almost constant.

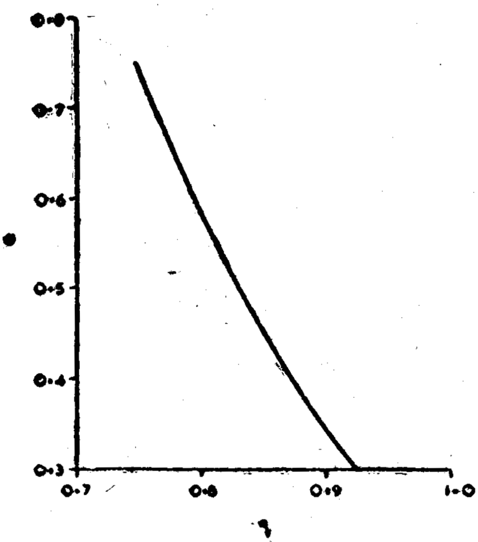

Fig. 1—Relation between e and q for transfer trajectories.

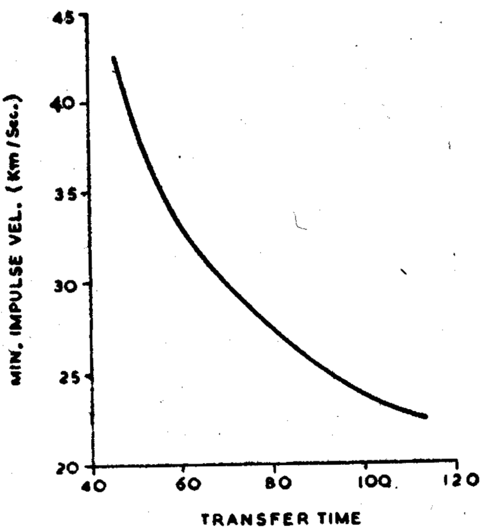

Fig. 2—Relation between minimum impulse velocity and transfer duration.

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