FLUID ROTATING IN THE PRESENCE OF A MAGNETIC FIELD

T. GOVINDARAJULU

Annamalai University, Annamalainagar

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The problem of rotating fluid recently considered by Thornley has been extended to electrically conducting fluids. It is found that the resonance effect encountered in the non-magnetic case is eliminated by the presence of a magnetic field.

Thornley[1] has investigated the flow of a viscous fluid in the semi-infinite region bounded by a plane when the fluid and the disc rotate like a solid body and additionally the disc performs non-torsional oscillations in its own plane. It is shown that oscillatory solution satisfying all the boundary conditions is impossible for the resonant case, i.e., when the frequency of the disc oscillation is twice the angular velocity of the basic rotation of the fluid. This particular behaviour has been analysed by treating an initial-value problem in which the oscillatory motion commences at time $t = 0$. The purpose of the present note is to show that in the case of an electrically conducting fluid with the presence of an axial magnetic field, this resonance effect is completely avoided. The effect of the magnetic field on the flow due to the impulsive start of the disc is also studied.

BASIC EQUATIONS AND SOLUTIONS BY LAPLACE TRANSFORM

Consider an infinite disc at $z = 0$ rotating in unision with the fluid in the region $z = 0$ with an angular velocity Ω about the z-axis. The fluid is assumed to be viscous, incompressible and electrically conducting. The plate, in addition to rotation, performs oscillations given by $q = u + iv = a\,e^{+int} + b\,e^{-int}$ in its own plane. Here a and b are complex constants while n may be taken to be real. The Navier-Stokes equations with respect to a set of coordinate system rotating with the fluid are

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} + 2\vec{\Omega} \times \vec{v} + \nabla \left\{ \frac{p}{\rho} - \tfrac{1}{2}\Omega^2 (x^2 + y^2) \right\} = \nu \nabla^2 \vec{v} \times \frac{\vec{j} \times \vec{B}}{\rho} \qquad (1)$$

$$\nabla \cdot \vec{v} = 0 \qquad (2)$$

where $\vec{v}$ is the velocity, p the pressure, ρ the density, ν the kinematic viscosity, $\vec{j}$ the current density and $\vec{B}$ the magnetic induction. Under the assumption that the magnetic Reynold number $R_m << 1$, the induced field is neglected. We also assume that the electric field

$\vec{E} \equiv 0$. With these assumptions the magnetic field $\vec{B} = (0, 0, B_0)$ where B_0 is the applied field fixed relative to the axes and the ponderomotive force $\frac{1}{\rho} \vec{j} \times \vec{B} = \left(-\frac{\sigma B_0^2 u}{\rho}, -\frac{\sigma B_0^2 v}{\rho}, 0 \right)$ where u, v are the velocity components parallel to the x-, y-axes. Taking all dependent variables to be functions of z and t only, (1) and (2) reduce[1] to

$$\frac{\partial u}{\partial t} - 2\,\Omega\, v = \nu \frac{\partial^2 u}{\partial z^2} - \frac{\sigma B_0^2}{\rho} u, \tag{3}$$

$$\frac{\partial v}{\partial t} + 2\,\Omega\, u = \nu \frac{\partial^2 v}{\partial z^2} - \frac{\sigma B_0^2}{\rho} v, \tag{4}$$

Putting $q = u + iv$, (3) and (4) can be combined as

$$\frac{\partial q}{\partial t} + 2\, i\, \Omega\, q = \nu \frac{\partial^2 q}{\partial z^2} - m^2 q, \tag{5}$$

where
$$m^2 = \frac{\sigma B_0^2}{\rho}.$$

The boundary conditions for the initial value problem are

$$\left.\begin{array}{l} \left.\begin{array}{l} (a)\ z = 0 : q = a\, e^{int} + b e^{-int} \\ (b)\ z \to \infty : \ q \to 0 \end{array}\right\} t > 0 \\ (c)\ q = 0 \text{ at } t = 0 \text{ for all } z \end{array}\right\} \tag{6}$$

Using the method of Laplace transform we can obtain the solution of (5) subject to (6) in the form

$$q = u + iv = \frac{a}{2}\, e^{int} \left[e^{\lambda_1 z/\nu^{\frac{1}{2}}} \operatorname{erfc}\left\{ \frac{z}{2(\nu t)^{\frac{1}{2}}} + \lambda_1 t^{\frac{1}{2}} \right\} + e^{-\lambda_1 z/\nu^{\frac{1}{2}}} \operatorname{erfc}\left\{ \frac{z}{2(\nu t)^{\frac{1}{2}}} - \lambda_1 t^{\frac{1}{2}} \right\} \right] + \frac{b}{2} e^{-int} \left[e^{\lambda_2 z/\nu^{\frac{1}{2}}} \operatorname{erfc}\left\{ \frac{z}{2(\nu t)^{\frac{1}{2}}} + \lambda_2 t^{\frac{1}{2}} \right\} + e^{-\lambda_2 z/\nu^{\frac{1}{2}}} \operatorname{erfc}\left\{ \frac{z}{2(\nu t)^{\frac{1}{2}}} - \lambda_2 t^{\frac{1}{2}} \right\} \right] \tag{7}$$

where
$$\lambda_1 = \{m^2 + i\,(2\,\Omega + n)\}^{\frac{1}{2}}, \quad \lambda_2 = \{m^2 + i\,(2\,\Omega - n)\}^{\frac{1}{2}}$$

Since the oscillatory solutions are reached at large times, we allow $t \to \infty$ and obtain

$$q = a\, e^{int - \lambda_1 z / \sqrt{\nu}} + b\, e^{-(int + \lambda_2 z / \sqrt{\nu})} \tag{8}$$

Now as $n \to 2\,\Omega$, (8) becomes

$$q = a\, e^{(int - \lambda_{10} z / \sqrt{\nu})} + b\, e^{-(int + m z / \sqrt{\nu})} \tag{9}$$

where $$\lambda_{10} = \lambda_1 \Big|_{n = 2\Omega}$$

This is the correct solution satisfying both the boundary conditions at $z = 0$ and at infinity, whereas when $m = 0$ (non-magnetic case) the condition at infinity is violated. Thornley also points out that the order in which the two limits namely, $t \to \infty$ and $n \to 2\,\Omega$, are taken cannot be interchanged when $z = 0\,\{(\nu t)^{\frac{1}{2}}\}$. This difficulty also does not arise in our case since λ_2 never vanishes.

Now we discuss the particular case when the disc is impulsively started in its own plane. The solution is directly derived from (7) by putting $n = 0$ and $a + b = c$ (a real constant, say). Hence we get

$$q = u + iv = \frac{c}{2}\left[e^{\lambda z/\nu^{\frac{1}{2}}} \operatorname{erfc}\left\{ \frac{z}{2(\nu t)^{\frac{1}{2}}} + \lambda t^{\frac{1}{2}} \right\} + e^{-\lambda z/\nu^{\frac{1}{2}}} \operatorname{erfc}\left\{ \frac{z}{2(\nu t)^{\frac{1}{2}}} - \lambda t^{\frac{1}{2}} \right\} \right] \tag{10}$$

where $$\lambda = \lambda_r + i\,\lambda_i = (m^2 + 2\,i\,\Omega)^{\frac{1}{2}}$$

To see how the boundary layer on the disc develops and finally settles down to a steady Ekman layer we write the solution for small and large times. Thus for small time t the expressions for u and v can be written as

$$\frac{u}{c} = \operatorname{erfc}\frac{z}{2(\nu t)^{\frac{1}{2}}} + \frac{z^2}{8\nu t}\left\{ 4m^2 t + t^2(m^2 - 4\Omega^2)\left(1 + \frac{z^2}{2\nu t}\right)\right\} \times$$

$$\times \operatorname{erfc}\frac{z}{2(\nu t)^{\frac{1}{2}}} - \frac{z}{2(\pi \nu t)^{\frac{1}{2}}}\left\{ 2m^2 t + \left(\frac{z}{2(\nu t)^{\frac{1}{2}}}\right)^2 t^2(m^4 - 4\Omega^2)\right\} e^{-z^2/4\nu t} \tag{11}$$

$$\frac{v}{c} = \Omega t\left(\frac{z^2}{2\nu t}\right)\left\{ 2 + m^2 t\left(1 + \frac{z^2}{2\nu t}\right)\right\} \operatorname{erfc}\frac{z}{2(\nu t)^{\frac{1}{2}}} - \frac{z\Omega t}{(\pi \nu t)^{\frac{1}{2}}} \times$$

$$\times \left(2 + \frac{z^2}{2\nu t} m^2 t\right) e^{-z^2/4\nu t} \tag{12}$$

and for large time they are

$$\frac{u}{c} = e^{-z\lambda_r/\nu^{\frac{1}{2}}} \cos(\lambda_i z/\nu^{\frac{1}{2}}) - \frac{z\, e^{-\left(\frac{z^2}{4\nu t} + m^2 t\right)}}{2R^2 t(\pi \nu t)^{\frac{1}{2}}}\{ m^2 \cos 2\Omega t - 2\Omega \sin 2\Omega t\} \tag{13}$$

$$\frac{v}{c} = -\left[e^{-z\lambda_r/\nu^{\frac{1}{2}}} \sin(\lambda_i z/\nu^{\frac{1}{2}}) - \frac{z\, e^{-\left(\frac{z^2}{4\nu t} + m^2 t\right)}}{2R^2 t(\pi \nu t)^{\frac{1}{2}}}\{ m^2 \sin 2\Omega t + 2\Omega \cos 2\Omega t\}\right] \tag{14}$$

where $$R = (m^4 + 4\,\Omega^2)^{\frac{1}{2}}$$

Just at the commencement of motion a Rayleigh layer unaffected by the magnetic field develops on the disc which is represented by the first term of (11). At large times a steady modified Ekman layer is formed whose thickness is of $0(\epsilon)$

where $$\epsilon = \left[\frac{2\nu}{m^2 + (m^4 + 2\,\Omega^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}}$$

This thickness decreases as the magnetic field strength increases. The oscillatory part given by the second term of each (12) and (13) die out with time and the increase in the field.

The tangential force per unit area on the disc is given by

$$P = p_{zx} + i\,p_{yz} = \rho\nu\left(\frac{\partial u}{\partial z} + i\,\frac{\partial v}{\partial z}\right)_{z=0}$$

$$= -\rho c\left(\frac{\nu}{\pi t}\right)^{\frac{1}{2}}\left\{e^{-\lambda^2 t} + \lambda\,(\pi t)^{\frac{1}{2}}\operatorname{erf}(\pi t^{\frac{1}{2}})\right\}$$

and, as $t \rightarrow \infty$ we have (corresponding to Ekman layers)

$$p_{zx} = -\rho c\left\{\frac{\nu(R + m^2)}{2}\right\}^{\frac{1}{2}}, \quad p_{yz} = -\rho c\left\{\frac{\nu(R - m^2)}{2}\right\}^{\frac{1}{2}}$$

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REFERENCE

1. THORNLEY, C., *Quart. J. Mech. Appl. Math.*, 21 (1968), 451.