TWO DIMENSIONAL CRACK PROBLEM

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An alternative method of finding distribution of stress in a long isotropic elastic cylinder containing a strip crack situated symmetrically on a diametral plane is presented here. The problem is reduced to the solution of a Fredholm integral equation of second kind by making use of suitable integral representation of complex potentials.

Recently Srivastav & Prem Narain[1] have given a method of determining distribution of stress in a long isotropic elastic cylinder containing a strip crack. They point out that the problem is equivalent to that of finding the stress distribution in a circular disc of homogeneous isotropic material containing a Griffith crack situated on a diameter. A solution of this two dimensional crack problem based on complex variable technique developed by England & Green[2] is given in this paper. The complex variable technique used here is a direct and a simpler way of solving this type of problem.

BASIC FORMULAE

The basic equations for two dimensional isotropic elasticity is quoted here. If r, θ are polar coordinates and

$$z = re^{i\theta}, \bar{z} = re^{-i\theta},$$

then

$$\mu\,(u_r + iu_\theta) = [\, k\Omega\,(z) - \bar{\Omega}\,(\bar{z}) + (z-\bar{z})\,\bar{\Omega}'\,(\bar{z}) - \bar{\chi}\,(\bar{z})\,]\, e^{i\theta} \tag{1}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 4\,[\,\Omega'\,(z) + \bar{\Omega}'\,(\bar{z})\,] \tag{2}$$

$$\sigma_{rr} + i\sigma_{r\theta} = 2\,[\,\Omega'\,(z) + \bar{\Omega}'\,(\bar{z})\,] - 2\,[\,(z-\bar{z})\,\bar{\Omega}''\,(\bar{z}) + \bar{\chi}'\,(\bar{z})\,]\, e^{-2i\theta} \tag{3}$$

$$\sigma_{\theta\theta} - i\sigma_{r\theta} = 2\,[\,\Omega'\,(z) + \bar{\Omega}'\,(\bar{z})\,] + 2\,[\,(z-\bar{z})\,\bar{\Omega}''\,(\bar{z}) + \bar{\chi}'\,(\bar{z})\,]\, e^{-2i\theta} \tag{4}$$

where (u_r, u_θ) are components of displacement and $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})$ are components of stress Also $k = 3-4\,\eta$ for plane strain and $k = (3-\eta)/(1+\eta)$ for generalized plane stress, η being Poisson's ratio. The above expressions are easily derived from those given by England & Green[2] by using the relations

$$\sigma_{rr} + \sigma_{\theta\theta} = \sigma_{xx} + \sigma_{yy}$$

$$u_r + u_\theta = e^{-i\theta}\,[\,u_x + u_y\,]$$

$$\sigma_{rr} - \sigma_{\theta\theta} + 2\,i\sigma_{r\theta} = e^{-2i\theta}\,[\,\sigma_{xx} - \sigma_{yy} + 2\,i\sigma_{xy}\,]$$

(see Green & Zerna[3])

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Polar coordinates are used. Suppose there is a line crack along the real axis between the points $0 \leqslant r \leqslant 1$ on the line $\theta = 0, \pi$ the region being bounded by the circle of radius $\rho >> 1$. We assume that the line crack is opened by equal and opposite normal pressure on each side of the crack. The following conditions must be satisfied on $\theta = 0$:

$$\left.\begin{aligned} \sigma_{\theta\theta}(r, 0) &= -f(r) - g(r) &&, 0 < r < 1, \theta = 0, \pi \\ \sigma_{r\theta}(r, 0) &= 0 &&, |r| \leqslant \rho \\ u_\theta(r, 0) &= 0 &&, 1 \leqslant |r| \leqslant \rho \end{aligned}\right\} \quad (5)$$

where $f(r)$ is even and $g(r)$ is odd function of r. We assume that $f(r)$ and $g(r)$ are sectionally continuous. If the circular boundary of the disc is free from traction, we have

$$\sigma_{rr}(\rho, \theta) = \sigma_{r\theta}(\rho, \theta) = 0, \qquad 0 \leqslant \theta \leqslant 2\pi \quad (6)$$

In order to solve the problem posed by (5) and (6), we assume that

$$\Omega(z) = \int_0^1 \frac{F(t) + zG(t)}{(z^2 - t^2)^{\frac{1}{2}}} dt + \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1} \quad (7)$$

and

$$\chi'(z) = \sum_{n=0}^{\infty} b_n z^n$$

where $F(t)$ and $G(t)$ are real continuous functions of t in the interval $0 \leqslant t \leqslant 1$.

If we examine the function on the line $\theta = 0$, we find that

$$(z^2 - t^2)^{\frac{1}{2}} = (r^2 - t^2)^{\frac{1}{2}}, \qquad r > t \text{ and } \theta \to \pm 0$$

and

$$(z^2 - t^2)^{\frac{1}{2}} = i\,(t^2 - r^2)^{\frac{1}{2}}, \qquad |r| < t, \quad \theta \to +0$$

$$(z^2 - t^2)^{\frac{1}{2}} = -i\,(t^2 - r^2)^{\frac{1}{2}}, \qquad |r| < t, \quad \theta \to -0.$$

Taking into account the above definitions of square root we see that the second and third boundary conditions of (5) are satisfied. The first condition of (5) yields the integral equations:

$$4 \frac{d}{dr} \int_0^r \frac{F(t)\, dt}{(r^2 - t^2)^{\frac{1}{2}}} + \sum_{n=0}^{\infty} (4 a_{2n} + 2 b_{2n}) r^{2n} = -f(r), \; 0 \leqslant r \leqslant 1$$

$$4\,\frac{d}{dr}\int_0^r \frac{r\,G(t)\,dt}{(r^2-t^2)^{\frac{1}{2}}}+\sum_{n=0}^{\infty}(4\,a_{2n+1}+2\,b_{2n+1})\,r^{2n+1}=-\,g(r),\;0\leqslant r\leqslant 1$$

which on inverting and simple adjustment of terms become

$$F(t)+\frac{t}{4}\left[4\,a_0+\sum_{n=0}^{\infty}(4\,n+8)\binom{\frac{1}{2}}{n+1}a_{2n+2}\,t^{2n+2}-\sum_{n=0}^{\infty}2\,(2\,n\,a_{2n}-b_{2n})\binom{\frac{1}{2}}{n}t^{2n}\right]=-\,\frac{t}{2\pi}\int_0^t\frac{f(r)\,dr}{(t^2-r^2)^{\frac{1}{2}}}\tag{8}$$

$$G(t)+\frac{t}{4}\left[3\,a_1+\sum_{n=0}^{\infty}2\,(2\,n+5)\binom{\frac{1}{2}}{n+2}a_{2n+3}\,t^{2n+2}-\sum_{n=0}^{\infty}2\left\{(2\,n+1)\,a_{2n+1}-b_{2n+1}\right\}\binom{\frac{1}{2}}{n+1}t^{2n}\right]=-\,\frac{1}{2\,\pi t}\int_0^t\frac{rg(r)\,dr}{(t^2-r^2)^{\frac{1}{2}}}\tag{9}$$

On the circular boundary

$$z=\rho e^{i\theta},\quad 0\leqslant\theta\leqslant\pi\text{ and }|\,t^2/z^2\,|<1,$$

hence

$$\Omega'(z)=\sum_{n=0}^{\infty}\left[a_n\,z^n-\binom{3/2}{n}\left\{A_{2n}\,z^{-2n}+B_{2n}\,z^{-2n-3}\right\}\right]$$

$$\Omega''(z)=\sum_{n=0}^{\infty}\left[na_n\,z^{n-1}+\binom{3/2}{n}\left\{(2\,n+2)\,A_{2n}\,z^{-2n-3}+(2\,n+3)\,B_{2n}\,z^{-2n-4}\right\}\right]$$

where

$$A_{2n}=\int_0^1 t^{2n}\,F(t)\,dt$$

$$B_{2n}=\int_0^1 t^{2n+2}\,G(t)\,dt$$

The boundary condition (6) after some manipulation yields following relation for the determination of coefficients a_n and b_n in terms of the functions $F(t)$ and $G(t)$.

$$2a_0 + a_1 \rho \cos\theta + \frac{2A_0}{\rho^2} + \frac{3B_0}{\rho^3}\cos\theta + \sum_{n=0}^{\infty}\left\{(na_n - b_n)\rho^n - na_{n+2}\rho^{n+2}\right\} \times$$

$$\times \cos(n+2)\theta + \sum_{n=0}^{\infty}(2n+4)\left\{\binom{3/2}{n+1}\frac{A_{2n+2}}{\rho^{2n+4}} - \binom{3/2}{n}\frac{A_{2n}}{\rho^{2n+2}}\right\}\cos(2n+2)\theta +$$

$$+ \sum_{n=0}^{\infty}(2n+5)\left\{\binom{3/2}{n+1}\frac{B_{2n+2}}{\rho^{2n+5}} - \binom{3/2}{n}\frac{B_{2n}}{\rho^{2n+3}}\right\}\cos(2n+3)\theta = 0,$$

$$-a_1\rho\sin\theta - \frac{3B_0}{\rho^3}\sin\theta + \sum_{n=0}^{\infty}\left\{(na_n - b_n)\rho^n - (n+2)a_{n+2}\rho^{n+2}\right\}\sin(n+2)\theta +$$

$$+ \sum_{n=0}^{\infty}\left\{\binom{3/2}{n}\frac{(2n+2)A_{2n}}{\rho^{2n+2}} - \binom{3/2}{n+1}\frac{(2n+4)A_{2n+2}}{\rho^{2n+4}}\right\}\sin(2n+2)\theta +$$

$$+ \sum_{n=0}^{\infty}\left\{\binom{3/2}{n}\frac{(2n+3)B_{2n}}{\rho^{2n+3}} - \binom{3/2}{n}\frac{(2n+5)B_{2n+2}}{\rho^{2n+5}}\right\}\sin(2n+3)\theta = 0,$$

from these expressions we get, for $n \geq 0$

$$a_{2n+2}\,\rho^{2n+2} = \binom{3/2}{n}\frac{(2n+3)A_{2n}}{\rho^{2n+2}} - \binom{3/2}{n+1}(2n+4)\frac{A_{2n+2}}{\rho^{2n+4}},$$

$$a_{2n+3}\,\rho^{2n+3} = \binom{3/2}{n}\frac{(2n+4)B_{2n}}{\rho^{2n+3}} - \binom{3/2}{n+1}(2n+5)\frac{B_{2n+2}}{\rho^{2n+5}},$$

$$(2na_{2n} - b_{2n})\,\rho^{2n} = (2n+2)^2\binom{3/2}{n}\frac{A_{2n}}{\rho^{2n+2}} - (2n+1)(2n+4)\binom{3/2}{n+1}\frac{A_{2n+2}}{\rho^{2n+4}}$$

$$\left\{(2n+1)a_{2n+1} - b_{2n+1}\right\}\rho^{2n+1} = (2n+3)^2\binom{3/2}{n}\frac{B_{2n}}{\rho^{2n+3}}$$

$$-\binom{3/2}{n+1}(2n+2)(2n+5)\frac{B_{2n+2}}{\rho^{2n+5}},$$

where
$$\binom{a}{n} = \frac{\Gamma(a+1)}{\Gamma(a+1-n)\,(n)!}\,.$$

Substituting the values of a_n and b_n in (8) and (9) we finally obtain Fredholm integral equations;

$$F(t) + t\int_0^1 F(x)\,K\left(\frac{xt}{\rho^2}\right)dx = -\frac{t}{2\pi}\int_0^t \frac{f(x)\,dx}{(t^2-x^2)^{\frac{1}{2}}} \tag{10}$$

$$G(t) + \int_0^1 x\,G(x)\,K_1\left(\frac{xt}{\rho^2}\right)dx = -\frac{1}{2\pi t}\int_0^t \frac{x\,g(x)\,dx}{(t^2-x^2)^{\frac{1}{2}}} \tag{11}$$

where

$$K\left(\frac{x\,t}{\rho^2}\right) = -\frac{3}{\rho^2} + \frac{x^2+t^2}{\rho^4}\sum_{n=0}^{\infty}(n+2)\binom{3/2}{n}\binom{3/2}{n+1}\left(\frac{x\,t}{\rho^2}\right)^{2n} - \frac{x^2t^2}{\rho^6}\sum_{n=0}^{\infty}(2n+4)^2\binom{\frac{1}{2}}{n+1}\binom{3/2}{n+1}\left(\frac{x\,t}{\rho^2}\right)^{2n},$$

$$K_1\left(\frac{x\,t}{\rho^2}\right) = -\frac{9\,x\,t}{\rho^4} + \frac{x\,t\,(x^2+t^2)}{\rho^6}\sum_{n=0}^{\infty}(2n+5)\binom{3/2}{n}\binom{3/2}{n+1}\left(\frac{x\,t}{\rho^2}\right)^{2n} - \frac{x^3t^3}{\rho^8}\sum_{n=0}^{\infty}2\,(2n+5)^2\binom{\frac{1}{2}}{n+1}\binom{3/2}{n+1}\left(\frac{x\,t}{\rho^2}\right)^{2n}.$$

For $\rho >> 1$, we have

$$K\left(\frac{x\,t}{\rho^2}\right) = -\frac{3}{\rho^2} + \frac{3\,(x^2+t^2)}{\rho^4} - \frac{12\,x^2t^2}{\rho^6} + 0\,(\rho^{-8})$$

$$K_1\left(\frac{x\,t}{\rho^2}\right) = -\frac{9\,x\,t}{\rho^4} + \frac{15x\,t\,(x^2+t^2)}{2\rho^6} + 0\,(\rho^{-8})$$

Now writing

$$\left.\begin{aligned} F(x) &= F_0(x) + F_1(x)\,\rho^{-2} + F_2(x)\,\rho^{-4} + \ldots\ldots\ldots\ldots \\ G(x) &= G_0(x) + G_1(x)\,\rho^{-2} + G_2(x)\,\rho^{-4} + \ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{12}$$

we get

$$F_0(t) = -\frac{t}{2\pi}\int_0^t \frac{f(x)\,dx}{(t^2-x^2)^{\frac{1}{2}}},\quad F_1(t) = 3\,t\int_0^1 F_0(x)\,dx$$

$$F_2(t) = 3\,t\int_0^1\left[F_1(x) - (x^2+t^2)\,F_0(x)\right]dx$$

$$F_3(t) = 3t \int_0^1 \left[F_2(x) - (x^2 + t^2) F_1(x) + 4x^2 t^2 F_0(x) \right] dx$$

$$G_0(t) = -\frac{1}{2\pi t} \int_0^t \frac{x\, g(x)\, dx}{(t^2 - x^2)^{\frac{1}{2}}}, \quad G_1(t) = 0$$

$$G_2(t) = 9t \int_0^1 x^2 G_0(x)\, dx$$

$$G_3(t) = -\frac{15t}{2} \int_0^1 x^2 (x^2 + t^2) G_0(x)\, dx$$

The normal displacement over the surface of a crack may be obtained from (1)

$$\mu\, u_\theta(r, 0) = -(1 + k) \int_{|r|}^1 \frac{F(t) + r\, G(t)}{(t^2 - r^2)^{\frac{1}{2}}}\, dt\,; \quad |r| \leqslant 1$$

Particular problems can now be examined, for example, if $f(r) = \frac{1}{2}p$, $g(r) = \frac{1}{2}p$, so that the crack is opened by uniform pressure over one half ($0 \leqslant r \leqslant 1$, $\theta = 0$), and zero pressure over remaining half ($0 \leqslant r \leqslant 1$, $\theta = \pi$), we have

$$F_0(t) = -\frac{pt}{8}, \; F_1(t) = -\frac{3pt}{16}, \; F_2(t) = \frac{3pt}{16}(t^2 - 1), \; F_3(t) = -\frac{3pt^3}{32} \quad (13)$$

and

$$G_0(t) = -\frac{p}{4\pi}, \; G_1(t) = 0, \; G_2(t) = -\frac{3pt}{4\pi}, \; G_3(t) = \frac{pt(3 + 5t^2)}{8\pi} \quad (14)$$

Hence,

$$\frac{\mu\, u_\theta(r, 0)}{1 + k} = \frac{p}{8}(1 - r^2)^{\frac{1}{2}} + \frac{3p}{16\rho^2}(1 - r^2)^{\frac{1}{2}} - \frac{3p}{16 r^2 \rho^4}(3 - 2r^2 - r^4)(1 - r^2)^{\frac{1}{2}} +$$

$$+ \frac{pr}{4\pi} \cosh^{-1} \frac{1}{|r|} + \frac{3pr}{4\pi\rho^4}(1 - r^2)^{\frac{1}{2}} + 0(\rho^{-6})$$

If we consider the case of a crack opened by a uniform pressure p, $G(t) = 0$ and the $F(t)$ terms take twice their values in (13). It is of interest in this case to derive the change in internal energy of the crack due to opening of crack. After simple calculation it is found to be equal to

$$\frac{\pi p^2 (1 + k)}{16\mu} \left[1 + \frac{3}{2\rho^2} + \frac{3}{4\rho^4} + 0(\rho^{-6}) \right] \quad (15)$$

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