TORSIONAL LOADINGS OF A PERFECTLY CONDUCTING ELASTIC HALF SPACE

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The problem of torsional oscillations of a semi-infinite perfectly conducting elastic medium with an applied magnetic field normal to the boundary due to (*i*) an impulsive load, (*ii*) step load, and (*iii*) sinusoidal load are solved. In the case of impulsive loading over a circular region, the duration of disturbance at a point is greater than in the case of a purely elastic material. The width of distribution is the same in both the cases.

The electromagnetic effects on stress wave propagation is of interest for a seismologist. Various wave problems are solved in magnetoelasticity and reviewed[1]. The problems of torsional vibrations of a semi-infinite medium under surface loadings and prescribed displacement on the surface are discussed[2,3], due to their applications in structural engineering, studies on earthquakes etc. In this paper the problem of torsional vibrations of a semi-infinite perfectly conducting medium when a large magnetic field is applied normal to the surface have been studied. In the case of impulsive loading over a circular region, the duration and region of disturbance at a point are greater than in the case of a purely elastic material. The width of disturbance is the same always. The expressions for displacement, stress components and secondary magnetic field near the epicentral line and near the surface are given. The results, when the loading is either a step function or sinusoidal in time, are also given. The results can be extended to anisotropic material without any difficulty[2].

STATEMENT OF THE PROBLEM AND SOLUTION

The cylindrical coordinate system (r, θ, z) is used such that the material medium is given by $z \geqslant 0$ and the origin coincides with the centre of application of the surface force. It is assumed that the displacement component, induced electric and magnetic fields are of the same order of smallness so that their products and derivatives are negligible. When the displacement currents are neglected in comparison with conduction currents, the linearised equations of motion are given by

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu) \operatorname{grad} \operatorname{div} \vec{u} + \mu \nabla^2 \vec{u} + (\vec{H} . \operatorname{grad}) \vec{h} - \operatorname{grad} (\vec{h} . \vec{H}) \tag{1}$$

The magnetic permeability of the medium is assumed to be unity approximately.

The axisymmetric displacement is given[2] by.

$$\vec{u} = u(r, z, t)\, \vec{i}_\theta \tag{2}$$

and the applied magnetic field is

$$\vec{H} = H_0 \, \vec{i_z} \tag{3}$$

where $\vec{i_r}$, $\vec{i_\theta}$, $\vec{i_z}$ are the unit vectors at a point in cylindrical coordinates.

The induced electric and magnetic fields are

$$\left.\begin{aligned} \vec{e} &= -\frac{\partial \vec{u}}{\partial t} \wedge \vec{H} = -H_0 \frac{\partial u}{\partial t} \vec{i_r} \\ \vec{h} &= \text{curl} \, (\vec{u} \wedge \vec{H}) = H_0 \frac{\partial u}{\partial z} \vec{i_\theta} \end{aligned}\right\} \tag{4}$$

Equations (1) and (4) give

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{H_0^2}{\mu} \frac{\partial^2 u}{\partial z^2} - \frac{\rho}{\mu} \frac{\partial^2 u}{\partial t^2} = 0 \tag{5}$$

On introducing the dimensionless parameters

$$u = aU, \; r = aR, \; z = aZ, \; t = Ta \left(\frac{\rho}{\mu}\right)^{\frac{1}{2}},$$

$$\alpha^2 = \frac{\mu}{\mu + H^2}$$

Equation (5) reduces to

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} - \frac{U}{R^2} + \frac{1}{\alpha^2} \frac{\partial^2 U}{\partial Z^2} - \frac{\partial^2 U}{\partial T^2} = 0 . \tag{6}$$

It is to be noted that α lies between zero and unity and it is very near to unity. $\alpha = 1$ refers to the case of a purely elastic material. Using Laplace's transform with respect to T and Hankel transform of order unity with respect to R defined by

$$\overline{U} = \int_0^\infty \int_0^\infty RU(R, Z, T) \, J_1(\xi R) \, e^{-sT} \, dR \, dT \tag{7}$$

equation (6) gives

$$\frac{d^2 \overline{U}}{dZ^2} - \alpha^2 (\xi^2 + s^2) \, \overline{U} = 0$$

for which the solution that remains finite as $Z \to \infty$, is

$$\overline{U} = A \, e^{-\alpha z (\xi^2 + s^2)^{\frac{1}{2}}} \tag{8}$$

where A is a constant of integration.

BOUNDARY CONDITIONS, STRESS COMPONENTS AND DISPLACEMENT

The boundary conditions are :

(*i*) continuity of the normal component of the magnetic field,

(*ii*) continuity of the tangential component of the electric field,

and (*iii*) continuity of the total stresses at the boundary.

Since unit magnetic permeability have been assumed, the first two conditions are satisfied identically and the third gives

$$\left.\begin{aligned} \sigma_{\theta z} &= P(r, t), \text{ the applied load at the boundary} \\ &= P(R, T) \\ \text{or} \quad \bar{\sigma}_{\theta z} &= \bar{P}(\xi, s) \end{aligned}\right\} \tag{9}$$

where $\bar{P}(\xi, s) = \int_0^\infty \int_0^\infty R\, P(R, T)\, J_1(\xi R)\, e^{-sT}\, dR\, dT$

which determines A to be

$$A = -\frac{\bar{P}(\xi, s)}{\alpha(\xi^2 + s^2)^{\frac{1}{2}}}$$

Hence the displacement components, stress components and the magnetic field are given by

$$U(R, Z, T) = -\frac{1}{2\pi i \mu \alpha} \int_0^\infty \int_{c-i\infty}^{c+i\infty} \frac{\xi \bar{P}(\xi, s)}{(\xi^2 + s^2)^{\frac{1}{2}}} \cdot e^{-\alpha z(\xi^2+s^2)^{\frac{1}{2}}} \times$$
$$\times J_1(\xi R)\, e^{sT}\, ds\, d\xi \tag{10}$$

$$\sigma_{z\theta} = \frac{1}{2\pi i} \int_0^\infty \int_{c-i\infty}^{c+i\infty} \xi \bar{P}(\xi, s)\, e^{-\alpha Z(\xi^2 + s^2)^{\frac{1}{2}}} J_1(\xi R)\, e^{sT}\, ds\, d\xi \tag{11}$$

$$\sigma_{r\theta} = \frac{1}{2\pi i \alpha} \int_0^\infty \int_{c-i\infty}^{c+i\infty} \frac{\xi^2 \bar{P}(\xi, s)}{(\xi^2 + s^2)^{\frac{1}{2}}} e^{-\alpha Z(\xi^2 + s^2)^{\frac{1}{2}}} J_2(\xi R)\, e^{sT}\, ds\, d\xi \tag{12}$$

$$h_\theta = H_0 \frac{\partial v}{\partial z} .$$

PARTICULAR CASE

Impulsive Surface Load : In this case, the surface load is

$$\left.\begin{aligned} P(r, t) &= R_1(r)\, \delta(t) \\ &= R_1(R)\, \delta\left\{a\, T\left(\frac{\rho}{\mu}\right)^{\frac{1}{2}}\right\} \end{aligned}\right\} \tag{13}$$

for which

$$\bar{P}(\xi, s) = \frac{1}{a}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \bar{R}_1(\xi)$$

where

$$\bar{R}_1(\xi) = \int_0^\infty R\, R_1(R)\, J_1(\xi R)\, dR.$$

The displacement is given by

$$U(R, Z, T) = -\frac{1}{\alpha\, a\, (\mu\, \rho)^{\frac{1}{2}}} \int_0^\infty \xi\, \bar{R}_1(\xi)\, J_1(\xi R)\, J_0(\xi y)\, d\xi, \qquad T > \alpha Z \qquad (14)$$

$$= 0 \qquad T < \alpha Z$$

where $y = (T^2 - \alpha^2 Z^2)^{\frac{1}{2}}$.

To have an idea of the electromagnetic effects, as the study is too close to that of reference 2, a particular case is studied in detail.

In this instance, $P(r, t) = Q\, r\, \delta(t) \qquad 0 < r < a$

$= 0 \qquad r > a$

i. e.,
$$P(R, T) = Q\, a\, R\, \delta\left\{ aT \left(\frac{\rho}{\mu}\right)^{\frac{1}{2}} \right\} \qquad 0 < R < 1$$

$$= 0 \qquad R > 1$$

The displacement in this case is obtained to be

$$U(R, Z, T) = -\frac{Q}{\alpha\, (\mu\, \rho)^{\frac{1}{2}}} \int_0^\infty J_2(\xi)\, J_1(\xi R)\, J_0(\xi y)\, d\xi \qquad (15)$$

$$= -\frac{Q}{\alpha\, (\mu\rho)^{\frac{1}{2}}} \begin{cases} 0 & R > y + 1 \\ 0 & R < y - 1 \\ R & R < 1 - y \\ \dfrac{1}{2\pi R}\Big\{ (R^2+1)(\pi - B_1) - \pi |R^2 - 1| \\ \quad - 2R \sin B_1 + 2|R^2 - 1| \times \\ \quad \times \tan^{-1}\left[\dfrac{R-1}{R+1} \tan \dfrac{B_1}{2}\right] - 2B_2 \Big\} \\ & |y-1| < R < y + 1 \end{cases}$$

where $\cos B_1 = \dfrac{R^2 + 1 - y^2}{2R}$, $\cos B_2 = \dfrac{1 + y^2 - R^2}{2R}$

In the region $R < 1 - y$, there is only a constant circumferential shear strain but no wave motion. When y is less than unity there will be a constant strain upto $R = 1 - y$ and from then onwards and upto $R = 1+y$, there will be time dependent disturbance. For $y > 1$, the region of disturbance is $y - 1 < R < y + 1$. In this case, the width of disturbance remains the same for all y. The values of y in this case are greater than its value corresponding to the purely elastic material. Hence the region of disturbance in the magnetoelastic case is greater than that of purely elastic material, because $\alpha < 1$. The duration of disturbance is given by

$$\left((R-1)^2 + \alpha^2 Z^2 \right)^{\frac{1}{2}} < T < \left((R+1)^2 + \alpha^2 Z^2 \right)^{\frac{1}{2}}$$

The wave arrives early and departs early in this case. The duration of disturbance in this case, t_m is given by

$$\left((R+1)^2 + \alpha^2 Z^2 \right)^{\frac{1}{2}} - \left((R-1)^2 + \alpha^2 Z^2 \right)^{\frac{1}{2}}$$

i. e.,

$$\frac{4\,R}{\left((R+1)^2 + \alpha^2 Z^2 \right)^{\frac{1}{2}} + \left((R-1)^2 + \alpha^2 Z^2 \right)^{\frac{1}{2}}}$$

whereas the duration of disturbance t_e, when the magnetic effects are absent is

$$\frac{4\,R}{\left((R+1)^2 + Z^2 \right)^{\frac{1}{2}} + \left((R-1)^2 + Z^2 \right)^{\frac{1}{2}}}$$

Hence the duration of disturbance in a magnetoelastic case is more.

The nonvanishing stress components are given by

$$\left.\begin{aligned} \sigma_{\theta z} &= \frac{Q\,\alpha\,Z}{(T^2 - \alpha^2 Z^2)^{\frac{1}{2}}} \left(\frac{\mu}{\rho} \right)^{\frac{1}{2}} \int_0^\infty J_2(\xi)\, J_1(\xi R)\, J_0(\xi y)\, d\xi \\ \sigma_{r\theta} &= -\frac{Q}{\alpha} \left(\frac{\mu}{\rho} \right)^{\frac{1}{2}} \int_0^\infty \xi\, J_2(\xi)\, J_2(\xi R)\, J_0(\xi y)\, d\xi\,. \end{aligned}\right\} T > \alpha Z \quad (16)$$

(*i*) *Near the surface* : The results for this case can be obtained by treating Z to be small and on making the simplifications, we get from (15) and (16)

$$U(R, Z, T) = -\frac{Q}{\alpha\, a\, (\mu \rho)^{\frac{1}{2}}} \int_0^\infty J_2(\xi)\, J_1(\xi R)\, J_0(\xi T)\, d\xi, \qquad T > \alpha Z$$

$$\sigma_{\theta Z} = \frac{Q\,\alpha\,Z}{T} \left(\frac{\mu}{\rho} \right)^{\frac{1}{2}} \int_0^\infty J_2(\xi)\, J_1(\xi R)\, J_0(\xi T)\, d\xi, \qquad T > \alpha Z$$

$$\sigma_{r\theta} = -\frac{Q}{a}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \int_0^\infty \xi\, J_2(\xi)\, J_2(\xi R)\, J_0(\xi T)\, d\xi, \qquad T > \alpha Z.$$

(*ii*) *Near the axis* : In this case, R is considered to be small and we get from (15) and (16)

$$U(R, Z, T) = -\frac{QR}{2\alpha\,(\mu\rho)^{\frac{1}{2}}} \int_0^\infty \xi\, J_2(\xi)\, J_0(\xi y)\, d\xi, \qquad T > \alpha Z$$

$$\sigma_{\theta Z}\,(R, Z, T) = \frac{Q\,\alpha\,R\,Z}{2\,(T^2 - \alpha^2 Z^2)^{\frac{1}{2}}}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \int_0^\infty \xi\, J_2(\xi)\, J_1(\xi y)\, d\xi, \qquad T > \alpha Z$$

$$\sigma_{r\theta}\,(R, Z, T) = -\frac{QR^2}{8\alpha}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \int_0^\infty \xi^3\, J_2(\xi)\, J_0(\xi y)\, d\xi, \qquad T > \alpha Z$$

The above quantities vanish when $T < \alpha z$

(*b*) *Step Loading* :

In this case, (9) is written as

$$P\,(R, T) = R_1(R)\, T_1(T)$$

where

$$T_1\,(T) = 1 \text{ for } T > 0$$
$$= 0 \text{ for } T < 0.$$

The displacement $U(R, Z, T)$ from (10) can be seen to be

$$U(\mathrm{R}, Z, T) = \int_0^{\mathrm{T}} U\,\Big|_{\text{impulsive loading}}\, dT.$$

(*c*) *Sinusoidal Loading*:

In this case, (9) is taken to be

$$P(R, T) = R_1(R)\, e^{i\omega T}$$

and the displacement (10) can be written as

$$U(R, Z, T) = R_T \sin\,(\,\omega T - \theta_T\,),$$

where $R_T \cos\theta_T = \displaystyle\int_0^{\mathrm{T}} \cos\,(\omega\, T)\; U\,\Big|_{\text{impulsive loading}}\, dT$;

$$R_T \sin\theta_T = \int_0^{\mathrm{T}} \sin\,(\omega\, T)\; U\,\Big|_{\text{impulsive loading}}\, dT$$

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