SOME INFINITE SUMMATION FORMULAE INVOLVING KAMPE de FERIET FUNCTION

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Two infinite summation formulæ for Kampé de Fériet function have been established and results are generalised by applying operational technique and method of finite mathematical induction. Various special cases are also obtained of which few are known.

The aim of this paper is to establish two infinite summation formulae for Kampé de Fériet function[1]. The results have been generalised by applying operational technique and method of finite mathematical induction. Some new interesting results have also been obtained. In what follows, (a_p) denotes the sequence of p parameters $a_1, a_2 \ldots\ldots a_p$ and $((a_p)_m)$ has the interpretation $\prod_{i=1}^{p} (a_i)_m$ and so on. Following formulae are required in the proof :

From Srivastava[2] [see p. 309, equations (3·2) and (3·3)]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; {}_{p+1}F_q \left[\begin{matrix} -n, (a_p), \\ (b_q), \end{matrix} x \right] \; {}_{r+1}F_s \left[\begin{matrix} -n, (\alpha_r), \\ (\beta_s), \end{matrix} y \right] \cdot z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n \, ((a_p)_n) \, ((\alpha_r)_n)}{n! \; (b_q)_n \; ((\beta_s)_n)} \times \left(\frac{xyz}{(1-z^2)} \right)^n \cdot {}_{p+1}F_q \left[\begin{matrix} \lambda+n, (a_p+n), \\ (b_q+n), \end{matrix} \frac{xz}{z-1} \right] \times$$

$$\times \; {}_{r+1}F_s \left[\begin{matrix} \lambda+n, (\alpha_r+n), \\ (\beta_s+n), \end{matrix} \frac{yz}{z-1} \right] \tag{1}$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; {}_{p+1}F_q \left[\begin{matrix} -n, (a_p), \\ (b_q), \end{matrix} x \right] \cdot {}_{r+1}F_{s+1} \left[\begin{matrix} -n, & (\alpha_r), \\ 1-\lambda-n, & (\beta_s), \end{matrix} y \right] \cdot z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{((a_p)_n) \, ((\alpha_r)_n)}{((b_q)_n) \, ((\beta_s)_n)} \; \frac{(-xyz)^n}{n!} \; {}_{p+1}F_q \left[\begin{matrix} \lambda, (a_p+n), \\ (b_q+n), \end{matrix} \frac{xz}{z-1} \right] \times$$

$$\times \; {}_rF_s \left[\begin{matrix} (\alpha_r+n), \\ (\beta_s+n), \end{matrix} yz \right] \tag{2}$$

(1) and (2) are valid for $p \leqslant q$, $r \leqslant s$ and $|x|$, $|y|$ and $|z|$ are so constrained that the two sides have a meaning.

From Srivastava[3] (see p.763)

$$\int_0^\infty e^{-zt}\, t^{\mu-1}\, {}_pF_q\left[\begin{matrix}(a_p), \\ (b_q),\end{matrix}\; xt\right] {}_rF_s\left[\begin{matrix}(\alpha_r), \\ (\beta_s),\end{matrix}\; yt\right] . K_\nu(zt) . dt$$

$$= \frac{\sqrt{\pi}\, . \Gamma(\mu+\nu) . \Gamma(\mu-\nu) .}{\Gamma(\mu+\frac{1}{2}) . (2z) .} F\left[\begin{matrix}\mu+\nu, \mu-\nu : (a_p) ; (\alpha_r) \\ \frac{2\mu+1}{2} : (b_q) ; (\beta_s)\end{matrix}\middle|\begin{matrix}\frac{x}{2z} \\ \frac{y}{2z}\end{matrix}\right] \tag{3}$$

provided $\quad p \leqslant q, r \leqslant s, Re(\mu) > | Re(\nu) |,$

$Re(z) > 0, \quad p < q, r < s$

$Re(z) > Re(x) + Re(y), \quad p = q, r = s$

In (1), replacing x by xt, y by yt, multiplying both sides by $e^{-zt} . t^{\mu-1} . K_\nu(zt)$, integrating w.r.t. to t between the limits 0 to ∞ and interchanging the order of integration and summation, we have

$$\sum_{n=0}^\infty \frac{(\lambda)_n z^n}{n!} \int_0^\infty e^{-zt} . (t)^{\mu-1} . K_\nu(zt) \times$$

$$\times\, {}_{p+1}F_q\left[\begin{matrix}-n, (a_p), \\ (b_q),\end{matrix}\; xt\right] {}_{r+1}F_s\left[\begin{matrix}-n, (\alpha_r), \\ (\beta_s),\end{matrix}\; yt\right] dt$$

$$= (1-z)^{-\lambda} \sum_{n=0}^\infty \frac{(\lambda)_n ((a_p))_n ((\alpha_r))_n}{n! ((b_q))_n ((\beta_s))_n} \left(\frac{xyz}{(1-z)^2}\right)^n \int_0^\infty e^{-zt} . (t)^{\mu+2n-1} \times$$

$$\times\, K_\nu(zt) . {}_{p+1}F_q\left[\begin{matrix}\lambda+n, (a_p+n), \\ (b_q+n),\end{matrix}\; \frac{xzt}{z-1}\right] {}_{r+1}F_s\left[\begin{matrix}\lambda+n, (\alpha_r+n), \\ (\beta_s+n),\end{matrix}\; \frac{yzt}{z-1}\right] dt$$

Evaluating the integrals on both sides by using (3) and replacing x by $2xz$, y by $2yz$ and μ, ν by $\frac{u+w}{2}$, $\frac{u-w}{2}$, we obtain the result

$$\sum_{n=0}^\infty \frac{(\lambda)_n}{n!} F\left[\begin{matrix}u, w : -n, (a_p) ; -n, (\alpha_r) \\ \frac{u+w+1}{2} : (b_q) : (\beta_s)\end{matrix}\middle|\begin{matrix}x \\ y\end{matrix}\right] z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^\infty \frac{(\lambda)_n\ ((a_p))_n\ ((\alpha_r))_n\ (u)_{2n}\ (w)_{2n}}{n! ((b_q))_n . ((\beta_s))_n \left(\frac{u+w+1}{2}\right)_{2n}} . \left(\frac{xyz}{(1-z)^2}\right)^n \times$$

$$\times F\left[\begin{matrix}u+2n, w+2n : \lambda+n, (a_p+n) ; \lambda+n, (\alpha_r+n), \\ \frac{u+w+1}{2}+2n ; (b_q+n) ; (\beta_s+n)\end{matrix}\middle|\begin{matrix}\frac{xz}{z-1} \\ \frac{yz}{z-1}\end{matrix}\right] \tag{4}$$

Similarly from (2), we have the result

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F\left[\begin{array}{l} u, w: -n, (a_p); \quad -n, (\alpha_r) \\ \frac{u+w+1}{2}: (b_q); 1-\lambda-n, (\beta_s) \end{array}\middle| \begin{array}{l} x \\ y \end{array}\right] . z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{((a_p))_n ((\alpha_r))_n (u)_{2n} (w)_{2n} (-x y z)^n}{((b_q)) ((\beta_n))_n \left(\frac{u+w+1}{2}\right)_{2n} n!} \times$$

$$\times F\left[\begin{array}{l} u+2n, w+2n: \lambda, (a_p+n); (\alpha_r+n) \\ \frac{u+w+1}{2}+2n: (b_q+n); (\beta_s+n) \end{array}\middle| \begin{array}{l} \frac{xz}{z-1} \\ yz \end{array}\right] \tag{5}$$

The results (4) and (5) are valid provided $p \leqslant q$, $r \leqslant s$, $Re\ (u+w) > |\ Re\ (u-w)\ |$ and $|\ x\ |$, $|y|$ and $|\ z\ |$ are so constrained that the two sides have a meaning.

The results (4) and (5) are generalised as

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F\left[\begin{array}{l} u, w, (c_\rho): -n, (a_p); -n, (\alpha_r) \\ \frac{u+w+1}{2}, (c'_\sigma): (b_q); (\beta_s) \end{array}\middle| \begin{array}{l} x \\ y \end{array}\right] . z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n ((a_p))_n ((\alpha_r))_n (u)_{2n} (w)_{2n} ((c_\rho))_{2n}}{n! ((b_q))_n ((\beta_s))_n \left(\frac{u+w+1}{2}\right)_{2n} ((c'_\sigma))_{2n}} \left[\frac{x y z}{(1-z)^2}\right]^n \times$$

$$\times F\left[\begin{array}{l} u+2n, w+2n, (c_\rho+2n): \lambda+n, (a_p+n); \lambda+n, (\alpha_r+n) \\ \frac{u+w+1}{2}+2n, \quad (c'_\sigma+2n): \quad (b_q+n); \quad (\beta_s+n) \end{array}\middle| \begin{array}{l} \frac{xz}{z-1} \\ \frac{yz}{z-1} \end{array}\right] \tag{6}$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F\left[\begin{array}{l} u, w, (c_\rho): \quad -n, (a_p); \quad -n, (\alpha_r) \\ \frac{u+w+1}{2}, (c'_\sigma): (b_q); 1-\lambda-n, (\beta_s) \end{array}\middle| \begin{array}{l} x \\ y \end{array}\right] . z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{((a_p))_n \ ((\alpha_r))_n \ (u)_{2n} \ (w)_{2n} \ ((c_\rho))_{2n} \ (-x y z)^n}{((b_q))_n \ ((\beta_s))_n \left(\frac{u+w+1}{2}\right)_{2n} ((c'_\sigma))_{2n} \, n!} \times$$

$$\times F\left[\begin{array}{l} u+2n, w+2n, (c_\rho+2n): \lambda, (a_p+n); (\alpha_r+n) \\ \frac{u+w+1}{2}+2n, (c'_\sigma+2n): (b_q+n); (\beta_s+n) \end{array}\middle| \begin{array}{l} \frac{xz}{z-1} \\ yz \end{array}\right] \tag{7}$$

provided $p \leqslant q, r \leqslant s$, $Re\,(u+w) > |\,Re\,(u-w)\,|$ and $|\,x\,|$, $|\,y\,|$ and $|\,z\,|$ are so constrained that the two sides have a meaning.

Clearly (6) is true for $\rho = \sigma = 0$ because of (4). For the proof of (6) by the method of finite mathematical induction, let us assume it true for some values of ρ and σ. Replacing x by xt, y by yt in (6) and multiplying both sides by $(t)^{c_\rho+1-1}$ and taking their Laplace transform w.r.t. t, we observe that ρ is replaced by $\rho + 1$. Again in (6) substituting $x = x/t$, $y = y/t$, multiplying both sides by $(t)^{-c'_\sigma+1}$ and taking the inverse Laplace transform we find that σ is replaced by $\sigma + 1$. Thus the induction on ρ and σ is complete. Hence (6) is proved.

Starting with (5) and making use of the above technique we arrive at the formula (7)

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(A) In (6), setting $p = q = r = s = 0$, $u = w + 1$, $c'_\sigma = w$, $\sigma - 1 = \rho$, replacing y by x and making use of the formula[1], we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; {}_{\rho+1}F_\rho \left[\begin{matrix} -2n, (c_\rho), \\ (c'_\rho), \end{matrix} \; x \right] \cdot z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n\,((c_\rho))_{2n}}{n!\,((c'_\rho))_{2n}} \left(\frac{x^2 z}{(1-z)^2} \right)^n {}_{\rho+1}F_\rho \left[\begin{matrix} 2\lambda + 2n, (c_\rho + 2n), \\ (c'_\rho + 2n) : \end{matrix} \; \frac{x z}{z-1} \right] \quad (8)$$

(*i*) Employing Rainville[4] formulae on left hand side of (8) with $x = 1$, $\rho = 1$ we have an interesting formula

$${}_3F_2 \left[\begin{matrix} \lambda, \dfrac{c_1' - c_1}{2}, \dfrac{c_1' - c_1 + 1}{2} \\ \dfrac{c_1'}{2}, \dfrac{c_1' + 1}{2} \end{matrix} \middle| \; z \right] = (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (c_1)_{2n}}{n!\,(c_1')_{2n}} \left(\frac{z}{(1-z)^2} \right)^n$$

$$\times \; {}_2F_1 \left[\begin{matrix} c_1 + 2n, 2\lambda + 2n, \\ c_1' + 2n, \end{matrix} \middle| \; \frac{z}{z-1} \right] \quad (9)$$

(*ii*) In (8), taking $\rho = 1$, replacing x by x/c_1, letting $c_1 \to \infty$ and employing Rainville[4] formulae, we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n\,(n+1)_n}{(c_1')_{2n}} \; L_{2n}^{c'_1 - 1}(x) \,.\, z^n = (1-z)^{\frac{c'_1}{2} - \lambda} \, (-x z)^{\frac{c'_1}{2}} \exp\left(\frac{xz}{2z-2} \right) \times$$

$$\times \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!\,(c'_1)_{2n}} \left(\frac{x}{z-1} \right)^n M_{k,\mu} \left(\frac{x z}{z-1} \right) \quad (10)$$

where $$k = \frac{c'_1}{2} - 2\lambda - n,\ \mu = \frac{c'_1 - 1}{2} + n .$$

(B) In (6) setting $p = q = r = s = 0$, $\rho = 1$, $\sigma = 2$, $u = w+1$ and $c'_1 = w$, we obtain a result involving Appell function[1] F_1 as

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n\,!} F_1 (c_1 , -n, -n : c'_1 , x, y) .\ z^n .$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n\ (c_1)_{2n}}{n\,!\ (c'_1)_{2n}} \left[\frac{x\,y\,z}{(1-z)^2}\right]^n \times$$

$$\times\ F_1 \left(c_1 + 2n,\ \lambda + n,\ \lambda + n\ ;\ c'_1 + 2n,\ \frac{x\,z}{z-1}\ ,\ \frac{y\,z}{z-1}\right) \qquad (11)$$

In (11) replacing y by x and employing the formula[1]

$$F_1 (\alpha , \beta , \beta^1 ; \nu, x , x) = {}_2F_1 \left(\begin{matrix} \alpha , \beta + \beta^1 \\ \nu, \end{matrix} \middle| x \right) ,$$

we again arrive at (8), with $\rho = 1$.

(C) In (6) substituting $p = r = 0$, $q = s = \sigma = \rho = 1$, $u = w+1$ and $c'_1 = w$, we obtain a known result[5].

(D) Taking $p = r = s = 0$, $q = \rho = \sigma = 1$, $u = w+1$ and $c'_1 = w$ in (7), we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n\,!} F_2 (c_1 , -n, -n\ ;\ b_1 ,\ 1 - \lambda - n\ ; x, y) .z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(c_1)_{2n}}{(b_1)_n\ n!} (-x\,y\,z)^n \sum_{r,\,s\,=\,0}^{\infty} \frac{(c_1 + 2n)_{r+s}\ (\lambda)_r}{(b_1 + n)_r\ r\,!\ s\,!} \left(\frac{x\,z}{z-1}\right)^r (y\,z)^s$$

Summing the series of s on right hand side and employing the definition of Horn's[6] function H_3, we have an interesting result

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n\,!} F_2 (c_1 , -n, -n\ ;\ b_1 ,\ 1 - \lambda - n : x, y) .z^n = (1 - z)^{-\lambda} (1 - y\,z)^{-c_1} \times$$

$$\times H_3 \left(c_1 , \lambda\ ;\ b_1\ ;\ \frac{-xyz}{(1-yz)^2}\ ,\ \frac{xz}{(z-1)\,(1-yz)} \right) \qquad (12)$$

Substituting $y = 0$ in (12), we have a known result[6].

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