ELASTIC RESPONSE OF A LARGE THIN PLATE TO AN IMPULSIVE DISTRIBUTION

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Small transverse symmetrical vibrations of a thin elastic plate submitted to Jacobi type pressure distribution have been described. The expressions for displacement and velocity are obtained in terms of Horn's function whereas for the central particle these entities are expressed in terms of Bessel functions. Numerical results are obtained for a particular case.

Sneddon[1] has considered the elastic response of a large thin plate due to a Gaussian pressure distribution varying with time, and Mitra[2] has discussed the elastic response due to an impulsive parabolic pressure confined to a finite region. In this paper, we consider a pressure distribution represented by a Jacobi polynomial. The physical importance and behaviour of a Jacobi type pressure distribution is discussed. The technique of integral transform is employed to study the problem. Numerical results are given for a particular case.

Jacobi polynomial of degree n and order α, β has already been defined[3] as

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n\,!}\ {}_2F_1\left[\begin{matrix} -n,\ 1+\alpha+\beta+n\ ; & \\ 1+\alpha\ ; & \dfrac{1-x}{2}\end{matrix}\right]$$

Jacobi polynomials form an orthogonal set over the interval $(-1, 1)$ with respect to the weight function $(1-x)^{\alpha}\ (1+x)^{\beta}$ for all values of α and $\beta > -1$. Gegenbauer, Ultraspherical, Tchebicheff and Legendre polynomials are the special cases of Jacobi polynomials. Again any function which is bounded and has a finite number of maxima and minima can be represented by a series of Jacobi polynomials. Hence by introducing Jacobi polynomials an investigation of elastic response due to an impulsive force of general nature is possible.

NOTATIONS

$z\,(r, t)$	=	$2A\,\rho h\delta\,(t-t_0)\,f\,(r)$, Intensity of the symmetrical acting force.
$2h$	=	Thickness of the plate.
ρ	=	Density of the material of the plate.
σ, E	=	Poisson's ratio and Young's modulus of the material of the plate.
A	=	Constant.
$w\,(r, t)$	=	Displacement of the central surface $z = 0$ of the plate.
b^2	=	$\dfrac{Eh^2}{3\rho\,(1-\sigma^2)}$
$\bar{X}\,(\xi)$	=	Hankel transform of order α of the functions.
$\delta\,(t)$	=	Dirac's delta function.

THE BEHAVIOUR OF $f(r)$

The type of pressure distribution considered here is one in which the radial dependence of the applied force is of the form

$$f(r) = \left(\frac{r}{a}\right)^{\alpha} \left(1 - \frac{r^2}{a}\right)^{\beta} P_n^{(\alpha,\,\beta)} \left(1 - 2\,\frac{r^2}{a^2}\right),$$

$$= \left(\frac{r}{a}\right)^{\alpha} \left(1 - \frac{r^2}{a^2}\right)^{\beta} \frac{(1+\alpha)_n}{n!}\; {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n\,; \\ 1+\alpha\,; \end{matrix} \; \frac{r^2}{a^2}\right],$$

$$= 0,\ (r \geqslant a)\,; \quad Re\,\alpha > -1,\ Re\,\beta > -1\,. \tag{1}$$

This type of pressure distribution is of a very general character and of great physical interest. We shall discuss a particular case for $n = 1$.

Let $n = 1$ then

$$f(r) = \left(\frac{r}{a}\right)^{\alpha} \left(1 - \frac{r^2}{a^2}\right)^{\beta} \left[1 + \alpha - (\alpha + \beta + 2)\,\frac{r^2}{a^2}\right]. \tag{2}$$

We shall have

$$f(r) = 0, \text{ when } r = a,\ r = a\sqrt{\frac{(\alpha+1)}{(\alpha+\beta+2)}}\,.$$

It should be noted that $f(r)$ increases as r decreases for any particular value of α; and

$$\left.\begin{aligned} f(r) &= +\,ve \qquad \text{when } 0 \ngtr r < a\sqrt{\frac{(\alpha+1)}{(\alpha+\beta+2)}} \\ &= -\,ve \qquad \text{when } a\sqrt{\frac{(\alpha+1)}{(\alpha+\beta+2)}} < r < a. \end{aligned}\right\} \tag{3}$$

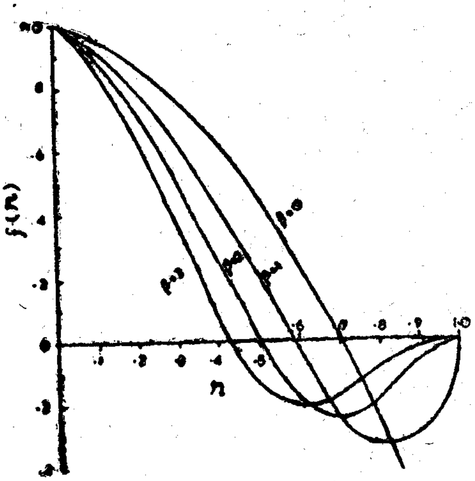

Fig. 1—Behaviour of $f(r)$ when $\alpha = 0$.

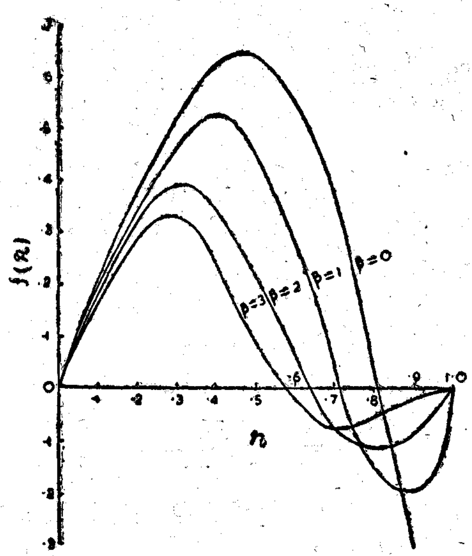

Fig. 2—Behaviour of $f(r)$ when $\alpha = 1$.

$$f(r) = \left(\frac{r}{a}\right)^{\alpha} \left(1 - \frac{r^2}{a^2}\right)^{\beta}, \text{ when } n = 0.$$

and
$$f(r) = 1, \qquad \text{when } n = 0, \quad \alpha = 0, \quad \beta = 0.$$

The physical significance of the negative sign of $f(r)$ implies that it is acting in the opposite direction. Thus by this type of representation it is possible to discuss the vibration of a plate subjected to the simultaneous action of the force acting in two opposite directions in different parts of the plate. The behaviour of $f(r)$ for $a = 1$; $\alpha = 0, 1$; $\beta = 0, 1, 2, 3$ and for different values of r is shown in Fig. 1 and 2.

SOLUTION OF THE PROBLEM

If we suppose that the intensity of the transverse force applied to the plate is $z(r, t) = 2A\,\rho h\delta\,(t - t_0)\,f(r)$, the fundamental differential equation[1] of the vibrations of a thin elastic plate will be of the form

$$b^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 w + \frac{\partial^2 \omega}{\partial t^2} = A\,\delta(t - t_0)\,f(r), \qquad (4$$

where the symbols have their usual meanings.

The boundary conditions are:

$$\left.\begin{aligned} \bar{w}\Big|_{r=a} &= 0. \\ \frac{\partial \bar{w}}{\partial t}\Big|_{r=a} &= 0. \end{aligned}\right\} \qquad (5)$$

Applying the Hankel transform of αth order in (4) we get

$$b^2\xi^4\,\bar{w} + \frac{\partial \bar{w}}{\partial t^2} = A\delta\,(t - t_0)\,\bar{f}\,(\xi) \qquad (6)$$

where $\bar{f}\,(\xi)$ is Hankel transform of $f(r)$ which is given by

$$\bar{f}\,(\xi) = \frac{2^{\beta}\,\Gamma\,(\beta + n + 1)}{a^{\beta - 1}\,n\,!\,\xi^{1+\beta}}\,J_{\alpha ++ 2n+1}(a\xi)$$

$$Re\,\alpha > -1, \quad Re\,\beta > -1. \qquad (7)$$

The solution of (6) satisfying conditions (5) is

$$\bar{w} = 0, \qquad t < t_0$$

$$= \frac{A2^{\beta}\,\Gamma\,(\beta + n + 1)}{b\,n\,!\,a^{\beta-1}\,\xi^{\beta+3}}\,J_{\alpha+\beta+2n+1}\,(a\xi)\,\sin b\xi^2(t - t_0),$$

$$t > t_0,\ Re\,\alpha > -1,\ Re\,\beta > -1 \qquad (8)$$

Thus

$$w = \frac{C_n}{b} \int_0^\infty \xi^{-\beta-2} J_{\alpha+\beta+2n+1}(a\xi)\, J_\alpha(r\xi) \sin b\xi^2 (t-t_0)\, d\xi ,$$

$$t > t_0 ,\ Re\,\alpha > -1,\ Re\,\beta > -1 \qquad (9)$$

and

$$\dot{w} = C_n \int_0^\infty \xi^{-\beta} J_{\alpha+\beta+2n+1}(a\xi)\, J_\alpha(r\xi) \cos b\xi^2 (t-t_0)\, d\xi ,$$

$$t > t_0 ,\ Re\,\alpha > -1 ,\ Re\,\beta > -1. \qquad (10)$$

Using Luke's[4] result to evaluate (9) and (10) and then applying transformation given in Rainville[3] and Bhonsle's[5] result, we get

$$w = \frac{D_n}{4b^n} I_m \left[e^{in\pi/2} \psi_2 (n, \alpha+\beta+2n+2; 1; ia^2/B,\ ir^2/B) \right],$$

$$t > t_0,\ Re\,\alpha > -1,\ Re\,\beta > -1,\ n > 0 , \qquad (11)$$

$$\dot{w} = \frac{D_n}{B} R \left[e^{in\pi/2} \psi_2 (n+1, \alpha+\beta+2n+2; 1; ia^2/B,\ ir^2/B) \right],$$

$$t > t_0,\ Re\,\alpha > -1,\ Re\,\beta > -1,\ n > -1 , \qquad (12)$$

where $D_n = \dfrac{A\,\Gamma(\beta+n+1)\, a^{2n+2}}{\Gamma(\alpha+\beta+2n+2)\, B^n}$, $B = 4b\,(t-t_0)$.

MOTION OF THE CENTRAL PARTICLE

Let r tends to zero in (9) and (10), we get

$$w = \frac{C_n}{b} \int_0^\infty \xi^{-\beta-2} J_{\alpha+\beta+2n+1}(a\xi) \sin b\xi^2 (t-t_0)\, d\xi , \qquad (13)$$

$$\dot{w} = C_n \int_0^\infty \xi^{-\beta} J_{\alpha+\beta+2n+1}(a\xi) \cos b\xi^2 (t-t_0)\, d\xi ,$$

$$t > t_0 ,\ Re\,\alpha > -1 ,\ Re\,\beta > -1 . \qquad (14)$$

Now using Luke's[4] and Slater's[6] , we get

$$w = \frac{\Gamma(n-\frac{1}{2})\, a^2 G_n}{8\, b^n} \times$$

$$\times \sum_{m=0}^{\infty} \frac{(2n-1)_m\ (-\alpha-\beta-2)_m}{m!\,(\alpha+\beta+2n+2)_m}\ J_{n+m-\frac{1}{2}}(a^2/2B) \sin\left[\frac{a^2}{2B} + (n+m)\frac{\pi}{2}\right],$$

$$t > t_0 ,\ Re\,\alpha > -1 ,\ Re\,\beta > -1 ,\ n > 0. \qquad (15)$$

and

$$\dot{w} = -\, 2G_n\, \Gamma\,(n + \tfrac{1}{2}) \times$$

$$\times \sum_{m=0}^{\infty} \frac{(2n+1)_m \;\; (-\alpha-\beta)_m}{m\,!\,(\alpha+\beta+2n+2)_m} J_{n+m+\frac{1}{2}}\,(a^2/2B)\;\cos\left[\frac{a^2}{2B} + (n+m+1)\,\frac{\pi}{2}\right]$$

$$t > t_0\,,\;\; Re\;\alpha > -1,\; Re\;\beta > -1,\; n > -1, \quad (1$$

where $$G_n = \frac{A\,(-1)^n\; 2^{2n}\;\;(a^2/2B)^{1/2}\,\Gamma\,(\beta+n+1)}{\Gamma\;(\alpha+\beta+2n+2)}\,.$$

DISCUSSION

Here we shall discuss the motion of the plate due to a particular type of pressure distribution which is obtained by taking

Case 1 : $\alpha = 0,\ \beta = n = 1.$

Here we have
$$f\,(r) = \left(1 - \frac{r^2}{a^2}\right)\left(1 - \frac{3r^2}{a^2}\right). \tag{17}$$

$$\therefore w = -\frac{1}{2Bb} \sum_{m=0}^{\infty} \frac{Aa^4 r^{2m}}{4m\,!\,B^m}\; {}_2F_1\left[\begin{matrix} -m, -m\,; \\ 5\,;\, \dfrac{a^2}{r^2}\end{matrix}\right] \sin\;(m+1)\,\frac{\pi}{2}, \tag{18}$$

$$\dot{w} = 2 \sum_{m=0}^{\infty} \frac{Aa^4 r^{2m}}{4m\,!\,B^m}\; {}_2F_1\left[\begin{matrix} -m\,, -m\,; \\ 5\,;\, \dfrac{a^2}{r^2}\end{matrix}\right] \cos\;(m+2)\,\frac{\pi}{2}. \tag{19}$$

Case 2 : $\alpha = 0,\ \beta = n = 1,\ r = a.$

Here we have
$$w = -\;\frac{Aa^4}{48Bb}\,R\left[{}_2F_2\left(\begin{matrix} 5/2,\; 3\,; \\ 5,\; 5\,; \end{matrix}\;\; \frac{i4a^2}{B}\right)\right], \tag{20}$$

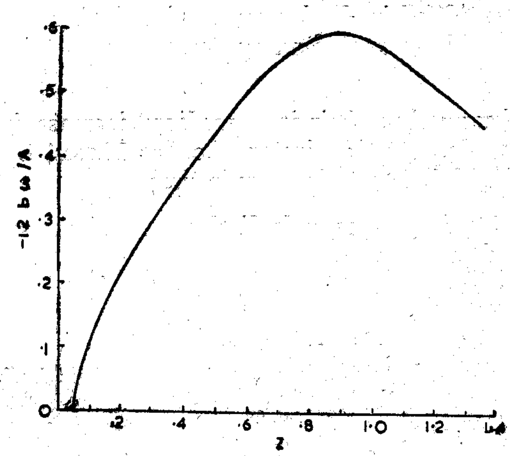

Fig. 3—Displacement curve for central particle.

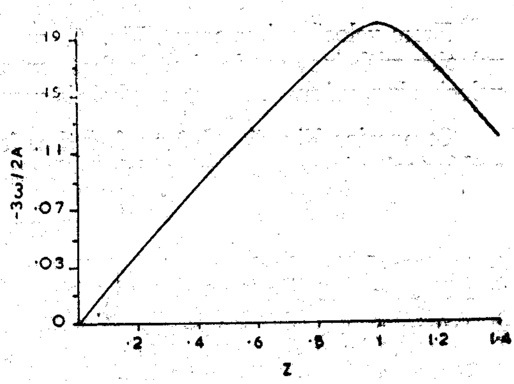

Fig. 4—Velocity curve for central particle.

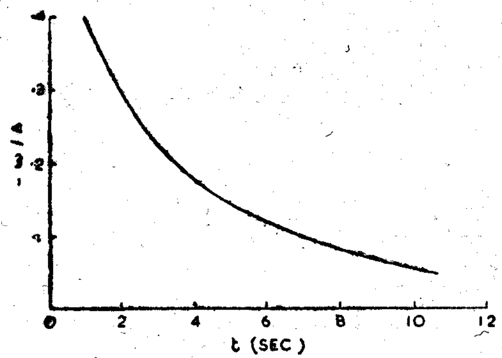

Fig. 5—Displacement versus time curve.

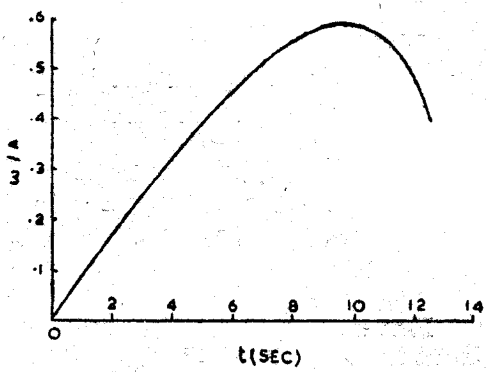

Fig. 6—Velocity versus time curve.

$$\dot{w} = -\frac{Aa^4}{12B^2} R \left[{}_3F_3 \left(\begin{matrix} 5/2, 3, 2\,; \\ 1, 5, 5\,; \dfrac{i4a^2}{B} \end{matrix} \right) \right]. \tag{21}$$

Case 3 : $\alpha = 0,\ \beta = n = 1,\ r = 0.$

Here we have from (15) and (16)

$$w = -\frac{Aa^3}{4b} \left(\frac{\pi}{B}\right)^{1/2} \Big[J_{1/2}(a^2/2B)\cos(a^2/2B) + \frac{3}{5} J_{3/2}(a^2/2B)\sin(a^2/2B) - $$

$$-\frac{1}{5} J_{5/2}(a^2/2B)\cos(a^2/2B) - 1/35\, J_{7/2}(a^2/2B)\sin(a^2/2B) \Big] \tag{22}$$

$$\dot{w} = -\frac{Aa}{3}\left(\frac{\pi}{B}\right)^{\frac{1}{2}} \left[J_{3/2}(a^2/2B)\cos(a^2/2B) + \frac{3}{5} J_{5/2}(a^2/2B)\sin(a^2/2B) \right] \tag{23}$$

Numerical values of w and $\dot{w}$ are plotted in Case 3 as shown in Fig 3–6.

CONCLUSION

We observe from Fig. 5 that their is a rapid fall in displacement with the increase of time upto first six seconds after which the fall is gradual tending towards zero with increase in time.

From velocity/time curve (Fig. 6), we observe that their is a medium increase in velocity with increase in time in first ten seconds. At this instant (i.e. $t = 10$ secs) the velocity is maximum. After it, their is a rapid fall in it with increase in time.

Comparing Fig. 5 and 6 we find that for the first ten seconds displacement decreases and velocity increases after which both start decreasing.

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