INTEGRATION OF SOME H-FUNCTIONS WITH RESPECT TO THEIR PARAMETERS

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Some integrals of H-functions with respect to their parameters have been established and employed to obtain some series of products of two H-functions.

MacRobert[1], Ragab[2–4] and Bajpai[5–6] have obtained a number of integrals of E-functions with respect to their parameters. Recently Bajpai[7] has obtained some integrals of G-functions with respect to their parameters. Anandani[8] has also obtained some integrals of H-functions with respect to their parameters which are the generalizations of the integrals of G-functions with respect to their parameters[7].

The object of this paper is to evaluate some integrals of Fox's H-functions with respect to their parameters and employ them to sum certain series of products of two H-functions.

The H-function introduced by Fox[9] will be represented and defined as follows :—

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,e_1),\ldots\ldots\ldots\ldots,(a_p,e_p)\\(b_1,f_1),\ldots\ldots\ldots\ldots,(b_q,f_q)\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)\,z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)}\,ds\,, \qquad (1)$$

where an empty product is interpreted as $1, 0\leqslant m\leqslant q, 0\leqslant n\leqslant p$; e's and f's are all positive; L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j-f_j s)$, $(j=1,\ldots\ldots,m)$ lie on the right hand side of the contour and those of $\Gamma(1-a_j+e_j s)$, $(j=1,\ldots\ldots,n)$ lie on the left-hand side of the contour.

According to Braaksma[10]

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]=0\,(|z|^e)\text{ for small }z,$$

where $\sum_1^p(e_j)-\sum_1^q(f_j)\leqslant 0$ and $e=Re\left(\dfrac{b_h}{f_h}\right)(h=1,\ldots\ldots\ldots\ldots,m)$

and $$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,e_p,)\\(b_q,f_q)\end{matrix}\right]=0\,(|z|^f)\text{ for large }z,$$

where $\sum_1^p(e_j)-\sum_1^q(f_j)<0,\ \sum_1^n(e_j)-\sum_{n+1}^p(e_j)+\sum_1^m(f_j)-\sum_{m+1}^q(f_j)\equiv\lambda>0,$

$|\arg z| < \frac{1}{2}\lambda\cdot\pi$ and $f = Re\left(\frac{a_i - 1}{e_i}\right)$ $(i = 1, \ldots\ldots\ldots, n)$.

In what follows for sake of brevity (a_p, e_p) denotes $(a_1, e_1), \ldots\ldots\ldots (a_p, e_p)$ and

$$\sum_1^p e_j - \sum_1^q f_j \equiv A, \sum_1^n e_j - \sum_{n+1}^p e_j + \sum_1^m f_j - \sum_{m+1}^q f_j \equiv B.$$

The relation between H and G-function is

$$H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_p, 1) \\ (b_q, 1) \end{matrix} \right.\right] = G_{p,q}^{m,n}\left[z \left| \begin{matrix} a_1, \ldots\ldots\ldots, a_p \\ b_1, \ldots\ldots\ldots, b_q \end{matrix} \right.\right] \tag{2}$$

Here we give the generalizations of known results[11], which will be required in the proofs which follow.

$$\int_0^1 x^{-\alpha}(1-x)^{\alpha-\beta-1} H_{p,q}^{m,n}\left[z x^\delta \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right.\right] dx$$

$$= \Gamma(\alpha-\beta)\, H_{p+1,q+1}^{m,n+1}\left[z \left| \begin{matrix} (\alpha, \delta), (a_p, e_p) \\ (b_q, f_q), (\beta, \delta) \end{matrix} \right.\right], \tag{3}$$

where $A < 0, B > 0, |\arg z| < \frac{1}{2} B\pi, Re(\alpha-\beta) > 0, Re\left(\delta\frac{b_j}{f_j} - \alpha\right) > -1\ (j = 1, \ldots, m)$

$$\int_1^\infty (1+x)^{-\lambda} \cdot x^{\lambda-\mu-1} H_{p,q}^{m,n}\left[z(1+x)^\delta \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right.\right]$$

$$= \Gamma(\lambda-\mu)\, H_{p+1,q+1}^{m+1,n}\left[z \left| \begin{matrix} (a_p, e_p), (\lambda, \delta) \\ (\mu, \delta), (b_q, f_q) \end{matrix} \right.\right], \tag{4}$$

where $A < 0, B > 0, |\arg z| < \frac{1}{2} B\cdot\pi, Re(\lambda-\mu) > 0,$

$$Re\left[\mu - \delta\frac{(a_j-1)}{e_j}\right] > 0\ (j = 1, \ldots\ldots\ldots, n).$$

INTEGRALS

The first integral to be established is

$$\frac{1}{2\pi i}\int_L \Gamma(s+k-\lambda)\,\Gamma(\lambda+\mu-s+\tfrac{1}{2})\,\Gamma(\lambda-\mu-s+\tfrac{1}{2}) z^s \times$$

$$\times H_{m+q+1, m+q+2}^{m+2, q}\left[z \left| \begin{matrix} (a_{1m+q} - \delta s, \delta), \ldots, (a_1 - \delta s, \delta), (1-\lambda-k, 1); \\ (\tfrac{1}{2}+\mu-\lambda, 1), (\tfrac{1}{2}-\mu-\lambda, 1), (b_1 - \delta s, \delta), \ldots\ldots, \\ (b_{m+q} - \delta s, \delta) \end{matrix} \right.\right] \times ds$$

$$= \pi^{-1/2}\,\Gamma(\tfrac{1}{2}+k-\mu)\,\Gamma(\tfrac{1}{2}+k+\mu)$$

$$\times H_{m+q+2, m+q+4}^{m+4, q}\left[\frac{z^2}{4} \left| \begin{matrix} (a_{m+q}, 2\delta), \ldots\ldots, (a_1, \delta), (1+k, 1), (1-k, 1); \\ (\tfrac{1}{2}, 1), (1, 1), (\tfrac{1}{2}+\mu, 1), (\tfrac{1}{2}-\mu, 1), (b_1, 2\delta), \ldots\ldots \\ \ldots\ldots\ldots, (b_{m+q}, 2\delta) \end{matrix} \right.\right] \tag{5}$$

provided $Re\,(a_r - b_r) > 0$, $(r = 1, \ldots\ldots, q)$, $Re\ (a_l - b_l) > 0$, $(l = 1, \ldots\ldots, m)$ $Re\ [\,\delta\,(\frac{1}{2} \pm \mu - \lambda) - a_r\,] > -1$, $(r = 1, \ldots\ldots\ldots, q + m)$, $Re\ (\lambda - k) < 0$, $Re\,\lambda > |\,Re\,\mu| - \frac{1}{2}$, L the path of integration being as in [11, p. 302, (29)] with loops, if necessary, to ensure that $(\lambda + \mu + \frac{1}{2})$ and $(\lambda - \mu + \frac{1}{2})$ are to the right of the contour and the H-functions exist.

Proof : By virtue of (3) and (4) the left hand side of (5) can be put in the form :

$$\frac{1}{2\pi i}\int_L \Gamma\,(s + k - \lambda)\,\Gamma\,(\lambda + \mu - s + \tfrac{1}{2})\,\Gamma\,(\lambda - \mu - s + \tfrac{1}{2})\,z^s \times$$

$$\times \prod_{j=1}^{q} \frac{1}{\Gamma\,(a_{m+j} - b_{m+j})} \prod_{j=1}^{q} \int_0^1 x_j^{-a_{m+j} + \delta s}\,(1 - x_j)^{a_{m+j} - b_{m+j} - 1}$$

$$\times \prod_{l=1}^{m} \frac{1}{\Gamma\,(a_l - b_l)} \prod_{l=1}^{m} \int_1^\infty (1 + x_l)^{-al + \delta s}\ xl^{a\,l - bl - 1}$$

$$\times H^{2,0}_{1,2}\left[z\,(x_j)^\delta\,(1 + x_l)^\delta \,\middle|\, \begin{matrix}(1 - \lambda - k, 1)\\ (\frac{1}{2} + \mu - \lambda, 1),\ (\frac{1}{2} - \mu - \lambda, 1)\end{matrix}\right] dx_j\,dx_l\,ds.$$

Now changing the order of integration and putting the first integral in the last, we get

$$\prod_{j=1}^{q} \frac{1}{\Gamma\,(a_{m+j} - b_{m+j})} \prod_{j=1}^{q} \int_0^1 x_j^{-a_{m+j}}\,(1 - x_j)^{a_{m+j} - b_{m+j} - 1} \times$$

$$\times \prod_{l=1}^{m} \frac{1}{\Gamma\,(a_l - b_l)} \cdot \prod_{l=1}^{m} \int_1^\infty (1 + x_l)^{-a_l}\ x_l^{\ a_l - b_l - 1}$$

$$\times H^{2,0}_{1,2}\left[z\,(x_j)^\delta\,(1 + x_l)^\delta \,\middle|\, \begin{matrix}(1 - \lambda - k, 1)\\ (\frac{1}{2} + \mu - \lambda, 1),\ (\frac{1}{2} - \mu - \lambda, 1)\end{matrix}\right] dx_j\,dx_l \times$$

$$\times \frac{1}{2\pi i}\int_L \Gamma\,(s + k - \lambda)\,\Gamma\,(\lambda + \mu - s + \tfrac{1}{2})\ \Gamma\ (\lambda - \mu - s + 1/2) \left\{ z\,x_j^\delta\,(1 + x_l)^\delta \right\}^s ds$$

Now substituting for the last integral[11] [ref. 11, p. 302, (29)], using (2) and [ref. 11, p. 435, (3)], the expression becomes

$$\Gamma\,(\tfrac{1}{2} + k - \mu)\,\Gamma\,(\tfrac{1}{2} + k + \mu) \prod_{j=1}^{q} \frac{1}{\Gamma\,(a_{m+j} - b_{m+j})} \prod_{j=1}^{q} \int_0^1 x_j^{-a_{m+j}}$$

$$(1 - x_j)^{a_{m+j} - b_{m+j} - 1} \times \prod_{l=1}^{m} \frac{1}{\Gamma\,(a_l - b_l)} \prod_{l=1}^{m} \int_1^\infty (1 + x_l)^{-a_l}\ x_l^{\ a_l - b_l - 1}$$

$$W_{k,\mu}\left\{ z x_j^{\ \delta}\,(1 + x_l)^\delta \right\}\ W_{-k,\mu}\left\{ z x_j^{\ \delta}\,(1 + x_l)^\delta \right\}\ dx_j \times dx.$$

On using (2) and [ref. 11, p. 443, (5)] we, have

$$\pi^{-\frac{1}{2}}\ \Gamma\left(\tfrac{1}{2}+k-\mu\right)\Gamma\left(\tfrac{1}{2}+k+\mu\right)\prod_{j=1}^{q}\frac{1}{\Gamma\left(a_{m+j}-b_{m+j}\right)}\cdot\prod_{j=1}^{q}\times$$

$$\times\int_{0}^{1} x_j^{-a_{m+j}}(1-x_j)^{a_{m+j}-b_{m+j}-1}\ \prod_{l=1}^{m}\frac{1}{\Gamma\left(a_l-b_l\right)}\ \prod_{l=1}^{m}\times$$

$$\times\int_{1}^{\infty}(1+x_l)^{-a_l}(x_l)^{a_l-b_l-1}$$

$$H_{2,4}^{4,0}\left[\frac{z^2}{4}x_j^{2\delta}(1+x_l)^{2\delta}\ \middle|\ \begin{matrix}(1+k,1),(1-k,1);\\ (\tfrac{1}{2},1),(1,1),(\tfrac{1}{2}+\mu,1),(\tfrac{1}{2}-\mu,1)\end{matrix}\right]dx_j\,dx_l$$

On applying (4) and (3) respectively the result (5) is obtained.

The second integral is

$$\frac{1}{2\pi i}\int_L \frac{\Gamma\left(\lambda+\mu-s+\tfrac{1}{2}\right)\Gamma\left(\lambda-\mu-s+\tfrac{1}{2}\right)}{\Gamma\left(\lambda+k-s+1\right)}\,z^s$$

$$\times H_{m+q+1,m+q+2}^{m+2,m+q+1}\left[z\ \middle|\ \begin{matrix}(1+k-\lambda,1),(a_{m+q}-\delta s,\delta),\ \ldots\ldots,(a_1-\delta s,\delta)\\ (\tfrac{1}{2}+\mu-\lambda,1),(\tfrac{1}{2}-\mu-\lambda,1),(b_1-\delta s,\delta),..(b_{m+q}-\delta s,\delta)\end{matrix}\right]\times ds$$

$$=\pi^{-1/2}\,\Gamma\left(\tfrac{1}{2}-k-\mu\right)\Gamma\left(\tfrac{1}{2}-k+\mu\right)\times$$

$$\times H_{m+q+2,m+q+4}^{m+4,q+m}\left[\frac{z^2}{4}\ \middle|\ \begin{matrix}(a_{m+q},2\delta),\ \ldots,(a_1,2\delta),(1+k,1),(1-k,1);\\ (\tfrac{1}{2},1),(1,1),(\tfrac{1}{2}+\mu,1),(\tfrac{1}{2}-\mu,1),(b_1,2\delta),\ldots,(b_{m+q},2\delta)\end{matrix}\right],\quad(6)$$

where $Re\,(a_r-b_r)>0,\ (r=1,\ \ldots\ldots,\ q),\ Re\ (a_l-b_l)>0,\ (l=1,\ \ldots\ldots,\ m),$

$Re\left[\ \delta\left(\tfrac{1}{2}\pm\mu-\lambda\right)-a_r\ \right]>-1,\ (r=1,\ \ldots,\ q+m)\ Re\,\lambda>|\ Re\,\mu\ |-\tfrac{1}{2},$

L the path of integration being as in [ref. 11, p. 302,(30)] with loops, if necessary, to ensure that $(\lambda+\mu+\tfrac{1}{2})$ and $(\lambda-\mu+\tfrac{1}{2})$ are to the right of the contour and the H-functions exist.

Proof: To prove this, we have applied the same procedure as in (5) and used [ref. 11, p. 302, (30); p 442, (9); 443, (5)] and(2).

The third integral is

$$\frac{1}{2\pi i}\int_L \Gamma\left(s-k-\lambda\right)\Gamma\left(\lambda+\mu-s+\tfrac{1}{2}\right)\Gamma\left(\lambda-\mu-s+\tfrac{1}{2}\right)z^s\times$$

$$\times H_{m+q+1,m+q+2}^{m+1,q+1}\left[z\ \middle|\ \begin{matrix}(1+k-\lambda,1),(a_{m+q}-\delta s,\delta),\ \ldots\ldots,(a_1-\delta s,\delta);\\ (\tfrac{1}{2}-\lambda+\mu,1),(b_1-\delta s,\delta),\ldots\ldots,(b_{m+q}-\delta s,\delta),(\tfrac{1}{2}-\lambda-\mu,1)\end{matrix}\right]ds$$

$$=\pi^{-1/2}\Gamma\left(\tfrac{1}{2}-k+\mu\right)\Gamma\left(\tfrac{1}{2}-k-\mu\right)$$

$$\times H_{m+q+2,m+q+4}^{m+3,q+1}\left[\frac{z^2}{4}\ \middle|\ \begin{matrix}(1+k,1),(a_{m+q},2\delta),\ \ldots\ldots,(a_1,2\delta),(1-k,1);\\ (\tfrac{1}{2},1),(1,1),(\tfrac{1}{2}+\mu,1),(b_1,2\delta),\ldots(b_{m+q},2\delta),(\tfrac{1}{2}-\mu,1)\end{matrix}\right]\quad(7)$$

where $Re\,(a_r - b_r) > 0, (r = 1, \ldots\ldots, q), Re\,(a_l - b_l) > 0, (l = 1, \ldots\ldots m),$

$$Re\left[\ \delta\,(\tfrac{1}{2} \pm \mu - \lambda\,) - a_r\right] > -1, (r = 1, \ldots\ldots\ldots, q + m), Re\,(\lambda + k) < 0,$$

$Re\,\lambda > |\,Re\,\mu\,| - \frac{1}{2}$; L the path of integration being as in [ref. 11, p. 302, (29)], with loops if necessary, to ensure that $\frac{1}{2} + \mu + \lambda$ and $\frac{1}{2} - \mu + \lambda$ are to the right of the contour and the H-functions exist.

Proof— To prove this integral, we have applied the same method as in (5) and used [ref. 11 p. 442, (7) p. 443, (3)],

The fourth integral is

$$\frac{1}{2\pi i}\int_L \frac{\Gamma\,(s - k - \lambda)\,\Gamma\,(\lambda + \mu - s + \frac{1}{2})}{\Gamma\,(\mu - \lambda + s + \frac{1}{2})}\,z^s \quad \times$$

$$\times\, H^{m+2,\,q+1}_{m+q+1,\,m+q+2}\left[z\,\middle|\,\begin{matrix}(1 + k - \lambda, 1), (a_{m+q} - \delta s, \delta), \ldots\ldots\ldots, (a_1 - \delta s, \delta);\\ (\frac{1}{2} + \mu - \lambda, 1), (\frac{1}{2} - \mu - \lambda, 1), (b_1 - \delta s, \delta), \ldots, (b_{m+q} - \delta s, \delta)\end{matrix}\right] ds$$

$$= \pi^{-1/2}\,\Gamma\,(\tfrac{1}{2} - k + \mu)\,\Gamma\,(\tfrac{1}{2} - k - \mu)\times$$

$$\times\, H^{m+3,\,q+1}_{m+q+2,\,m+q+4}\left\{\frac{z^2}{4}\,\middle|\,\begin{matrix}(1 + k, 1), (a_{m+q}, 2\delta), \ldots\ldots\ldots\ldots, (a_1, 2\delta), (1 - k, 1);\\ (\frac{1}{2}, 1), (1, 1), (\frac{1}{2} + \mu, 1), (b_1, 2\delta), \ldots\ldots\ldots\ldots,\\ (b_{m+q}, 2\,\delta), (\frac{1}{2} - \mu, 1)\end{matrix}\right\}, \quad (8)$$

where $Re\,(a_r - b_r) > 0, (r = 1, \ldots\ldots\ldots q), Re\,(a_l - b_l) > 0, (l = 1, \ldots\ldots\ldots m),$

$$Re\left[\,\delta\left(\tfrac{1}{2} \pm \mu - \lambda\right) - a_r\right] > -1, (r = 1, \ldots\ldots\ldots\ldots, q + m), Re\,(\lambda + k) < 0,$$

$Re\,\lambda > |\,Re\,\mu\,| - \frac{1}{2}$, L the path of integration being as in [ref. 11, p. 302 ,(31)] with loops, if necessary to ensure that $\frac{1}{2} + \mu + \lambda$ and $\frac{1}{2} - \mu + \lambda$ are to the right of the contour and H-functions exist.

Proof: The integral can be established by applying the same procedure as in (5) and using [ref. 11, p. 302, (31); p. 442, (9); p. 443, (3)] and (2).

SUMMATION OF SERIES

The first summation is

$$\sum_{r=0}^{\infty} \frac{(1 - b_2)_r}{r\,!}\, H^{2,\,1}_{1,\,2}\left[z\,\middle|\,\begin{matrix}(a_1, \sigma)\\ (b_1 + r, \sigma), (b_2, \sigma)\end{matrix}\right]\times$$

$$\times\, H^{2,\,1}_{2,\,3}\left[z\,\middle|\,\begin{matrix}(1 + a_2 - b_1 - b_2, \sigma), (1 + a_1 - b_1 - b_2, \sigma)\\ (1 - b_2, \sigma), (1 - b_1, \sigma), (a_2 - b_1 - r, \sigma)\end{matrix}\right]$$

$$= \pi^{-1/2}\,\sigma^{-2}\,\Gamma\,(1 - a_1 + b_1)\,\Gamma\,(1 - a_1 + b_2)\times$$

$$+ H^{4,\,1}_{3,\,5}\left[\tfrac{1}{4}z^{2/\sigma}\,\middle|\,\begin{matrix}(1 + a_2 - b_2, 2), \left(\frac{3 - 2\,a_1 + b_1 + b_2}{2}, 1\right), \left(\frac{1 + 2\,a_1 - b_1 - b_2}{2}, 1\right),\\ (\frac{1}{2}, 1), (1, 1), \left(\frac{1 + b_1 - b_2}{2}, 1\right), \left(\frac{1 + b_2 - b_1}{2}, 1\right), (a_2, 1)\end{matrix}\right], \quad (9)$$

where $|\arg z| < \sigma\pi$, $Re\,(1-a_1+b_1)>0$, $Re\,(1-a_1+b_2)>0$, and the H-functions exist

Proof: To prove (9), substituting on the left from (1), we have :

$$\sum_{r=0}^{\infty} \frac{(1-b_2)_r}{r!} \frac{1}{2\pi i} \int_L \Gamma(b_1+r-\sigma s)\,\Gamma(b_2-\sigma s)\,\Gamma(1-a_1+\sigma s)\, z^s\, ds \times$$

$$\times \frac{1}{2\pi i} \int_L \frac{\Gamma(1-b_2-\sigma w)\,\Gamma(1-b_1-\sigma w)\,\Gamma(b_1+b_2-a_2+\sigma w)\, z^w}{\Gamma(1+a_1-b_1-b_2-\sigma w)\,\Gamma(1+b_1-a_2+r+\sigma w)} \,.\, dw$$

replacing s by $\frac{s}{\sigma}$ and w by $\frac{w}{\sigma}$ and changing the order of integration and summation,

in view of [ref. 12, p. 500], the expression becomes

$$\sigma^{-2}\, \frac{1}{2\pi i} \int_L \Gamma(b_1-s)\,\Gamma(b_2-s)\,\Gamma(1-a_1+s)\, z^{s/\sigma}\, ds \times$$

$$\times \frac{1}{2\pi i} \int_L \frac{\Gamma(1-b_2-w)\,\Gamma(1-b_1-w)\,\Gamma(b_1+b_2-a_2+w)}{\Gamma(1+a_1-b_1-b_2-w)\,\Gamma(1+b_1-a_2+w)}$$

$$\times {}_2F_1 \left(\begin{matrix} b_1-s,\, 1-b_2\,;1 \\ (1+b_1-a_2+w \end{matrix} \right) z^{w/\sigma}\, dw$$

Now applying Gauss's theorem and substituting $a_1 = 1-k+\lambda$, $a_2 = {}^3/_2-\gamma+\lambda+\mu$, $b_1 = \lambda+\mu+\frac{1}{2}$ and $b_2 = \lambda-\mu+\frac{1}{2}$, it reduces to

$$\sigma^{-2}\, \frac{1}{2\pi i} \int_L \Gamma(s+k-\lambda)\,\Gamma(\lambda+\mu-s+\tfrac{1}{2})\,\Gamma(\lambda-\mu+\tfrac{1}{2}-s) \times$$

$$\times H_{2,3}^{2,1}\left[z^{1/\sigma} \left| \begin{matrix} (2-\gamma+2\mu-s,1),(1-k-\lambda,1) \\ (\mu-\gamma+\frac{1}{2},1),(\frac{1}{2}-\mu-\lambda,1),({}^3/_2-\gamma+\lambda+\mu-s,\,1) \end{matrix} \right. \right] z^{s/\sigma}\, ds,$$

using (5) with z replaced by $z^{1/\sigma}$, $\delta=1$, $q=1$, $m=0$, $a_1 = 2-\gamma+2\mu$,

$b_1 = {}^3/_2-\gamma+\lambda+\mu$, we get the expansion equal to

$$\sigma^{-2}\, \pi^{-1/2}\, \Gamma(\tfrac{1}{2}+k-\mu)\,\Gamma(\tfrac{1}{2}+k+\mu)\times$$

$$\times H_{3,5}^{4,1}\left[\tfrac{1}{4}\, z^{2/\sigma} \left| \begin{matrix} (2-\gamma+2\mu,2),(1+k,1),(1-k,1) \\ (\frac{1}{2},1),(1,1),(\frac{1}{2}+\mu,1),(\frac{1}{2}-\mu,1),({}^3/_2-\gamma+\lambda+\mu,2) \end{matrix} \right. \right].$$

Now substituting the values of k, γ, μ in terms of a_1. a_2, b_1, and b_2, we get the result.

The second summation is :

$$\sum_{r=0}^{\infty} \frac{(1-b_2)_r}{r!} H_{1,2}^{2,1}\left[z \left| \begin{matrix} (a_1, \sigma) \\ (b_1+r, \sigma), (b_2, \sigma) \end{matrix} \right.\right] \times$$

$$\times H_{2,3}^{1,2}\left[z \left| \begin{matrix} (1+a_1-b_1-b_2, \sigma), (1+a_2-b_1-b_2, \sigma) \\ (1-b_2, \sigma), (b_1, \sigma), (a_2-b_1-r, \sigma) \end{matrix} \right.\right]$$

$$= \sigma^{-2} \pi^{-1/2} \Gamma(1-a_1+b_1) \Gamma(1-a_1+b_2) \times$$

$$\times H_{3,5}^{3,2}\left[\frac{1}{4} z^{2/\sigma} \left| \begin{matrix} \left(\frac{1+2a_1-b_1-b_2}{2}, 1\right), (1+a_2-b_2, 2), \left(\frac{3-2a_1+b_1+b_2}{2}, 1\right); \\ (\tfrac{1}{2}, 1), (1, 1), \left(\frac{1+b_1-b_2}{2}, 1\right), (a_2, 2), \left(\frac{1-b_1+b_2}{2}, 1\right) \end{matrix} \right.\right], \tag{10}$$

provided $|\arg z| < \sigma\pi$, $Re(1+b_1-a_2) > 0$, $Re(1-a_1+b_1) > 0$, $Re(1-a_1+b_2) > 0$, and the H-functions exist.

Proof—This series can be established by applying the same procedure as in (9), substituting $a_1 = 1+k+\lambda$, $a_2 = {}^3/_2 - \gamma + \lambda + \mu$, $b_1 = \lambda + \mu + \frac{1}{2}$, $b_2 = \lambda - \mu + \frac{1}{2}$ and using (7)

The third summation is

$$\sum_{r=0}^{\infty} \frac{(1-b_1)_r}{r!} H_{2,3}^{1,2}\left[z \left| \begin{matrix} (1+a_1-2b_1, \sigma), (1+a_2-b_1-b_2, \sigma) \\ (1-b_1, \sigma), (1-b_2, \sigma), (a_1-b_1-r, \sigma) \end{matrix} \right.\right] \times$$

$$\times H_{1,2}^{2,1}\left[z \left| \begin{matrix} (a_2, \sigma) \\ (b_1+r, \sigma), (b_2, \sigma) \end{matrix} \right.\right]$$

$$= \sigma^{-2} \pi^{-\frac{1}{2}} \Gamma(1-a_2+b_1) \Gamma(1-a_2+b_2) \times$$

$$\times H_{3,5}^{3,2}\left[\tfrac{1}{4} z^{2/\sigma} \left| \begin{matrix} \left(\frac{1+2a_2-b_1-b_2}{2}, 1\right), (1+a_1-b_1, 2), \left(\frac{3-2a_2+b_1+b_2}{2}, 1\right); \\ (\tfrac{1}{2}, 1), (1, 1), \left(\frac{1-b_1+b_2}{2}, 1\right), (a_1, 2), \left(\frac{1+b_1-b_2}{2}, 1\right) \end{matrix} \right.\right], \tag{11}$$

where $|\arg z| < \sigma\pi$, $Re(1+b_1-a_1) > 0$, $Re(1-a_2+b_1) > 0$, $Re(1-a_2+b_2) > 0$ and the H-functions exist.

Proof : The series can be established by applying the same procedure as in (9), substituting $a_1 = {}^3/_2 - \mu - \lambda - \gamma$, $a_2 = 1+k-\lambda$, $b_1 = \frac{1}{2} - \mu - \lambda$, $b_2 = \frac{1}{2} + \mu - \lambda$ and using (8).

The fourth summation is:

$$\sum_{r=0}^{\infty} \frac{(1-b_1)_r}{r!} H_{2,3}^{2,1}\left[z \middle| \begin{matrix} (1-2b_1+a_1, \sigma), (a_2-b_1-b_2, \sigma) \\ (1-b_1, \sigma), (-b_2, \sigma), (a_1-b_1-r, \sigma) \end{matrix} \right] \times$$

$$\times H_{1,2}^{2,1}\left[z \middle| \begin{matrix} (a_2, \sigma) \\ (b_1+r, \sigma), (1+b_2, \sigma) \end{matrix} \right]$$

$$= \sigma^{-2} \pi^{-1/2} \Gamma(1-a_2+b_1) \Gamma(2-a_2+b_2) \times$$

$$\times H_{3,5}^{4,1}\left[\tfrac{1}{4} z^{2/\sigma} \middle| \begin{matrix} (1+a_1-b_1, 2), \left(\frac{2a_2-b_1-b_2}{2}, 1\right), \left(\frac{4-2a_2+b_1+b_2}{2}, 1\right); \\ (\tfrac{1}{2}, 1), (1,1), \left(\frac{2-b_1+b_2}{2}, 1\right), \left(\frac{b_1-b_2}{2}, 1\right), (a_1, 2) \end{matrix} \right], \quad (12)$$

where $|\arg z| < \sigma\pi$, $Re(1-a_1+b_1) > 0$, $Re(2-a_2+b_2) > 0$, $Re(1-a_2+b_1) > 0$ and the H-functions exist.

Proof: This series can be established by applying the same procedure as in (9) $a_1 = 3/2 - \mu - \lambda - \gamma$, $a_2 = 1-k-\lambda$, $b_1 = \frac{1}{2} - \mu - \lambda$, $b_2 = -\frac{1}{2} + \mu - \lambda$ and using (6).

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