BENDING OF AN ISOTROPIC COMPRESSIBLE RECTANGULAR BLOCK INTO A PARABOLIC SHELL

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The problem of bending of an isotropic compressible rectangular block into a parabolic shell has been solved. It is shown that the deformation can be maintained by applying forces and couples to the edges of the shell only.

Bending of rectangular plates into cylindrical shells has been considered by various authors[1-3] on the basis of linear stress strain relations by referring the components of finite strain to the strained state of the body. The exact solutions obtained for compressible bodies have not been very large. In the present paper a solution is obtained for the problem of bending of an isotropic compressible rectangular bolck into a parabolic shell. The strain tensor has been calculated directly from the metric tensors of the strained and unstrained states of the body without any reference to the displacements. We use Seth's stress-strain relations[2]

$$\sigma_{ij} = \lambda\, I\, \delta_{ij} + 2\, \mu\, e_{ij} \tag{1}$$

The notation used by Green & Zerna[4] have been followed.

DEFORMATION OF THE BLOCK

Let a rectangular block in the undeformed state be bounded by the planes $x_1 = a_1$, $x_1 = a_2\ (a_2 > a_1)$, $x_2 = \pm\, b$ and $x_3 = \pm\, d$. Then it is bent into a part of parabolic shell whose internal and external boundaries are the confocal parabolas.

$$x_1 = \frac{1}{2}\left(\zeta^2 - \eta^2\right), x_2 = \zeta\,\eta, \zeta = \zeta_i\,, i = 1, 2 \tag{2}$$

with the edges $\eta = \pm\,\alpha$. Let y_i-axes coincide with the x_i-axes and the curvilinear coordinates θ_i in the deformed state be a system of orthogonal curvilinear coordinates (ζ, η, z); so that

$$y_1 = \frac{1}{2}\left(\zeta^2 - \eta^2\right),\quad y_2 = \zeta\,\eta, y_3 = z \tag{3}$$

Since the deformation is symmetric about the x_1-axis, and supposing that there is no extension in the x_3-direction, we get

$$x_1 = f(\zeta),\ x_2 = F(\eta),\ x_3 = z \tag{4}$$

The metric tensors for the strained and unstrained states are given by

$$G_{ij} = \begin{bmatrix} \zeta^2+\eta^2 & 0 & 0 \\ 0 & \zeta^2+\eta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5)$$

$$g_{ij} = \begin{bmatrix} f'^2 & 0 & 0 \\ 0 & F'^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (6)$$

where $f' = \dfrac{df}{d\zeta}$ and $F' = \dfrac{dF}{d\eta}$. Then

$$2\gamma_{ij} = \begin{bmatrix} \zeta^2+\eta^2-f'^2 & 0 & 0 \\ 0 & \zeta^2+\eta^2-F'^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (7)$$

The physical components of stress are then given by

$$\left.\begin{aligned} 2\,\sigma_{11} &= 2(\lambda+\mu) - \frac{\lambda F'^2}{\zeta^2+\eta^2} - \frac{(\lambda+2\mu) f'^2}{\zeta^2+\eta^2} \\ 2\,\sigma_{22} &= 2(\lambda+\mu) - \frac{(\lambda+2\mu) F'^2}{\zeta^2+\eta^2} - \frac{\lambda f'^2}{\zeta^2+\eta^2} \\ 2\,\sigma_{33} &= \left[2 - \frac{f'^2+F'^2}{\zeta^2+\eta^2}\right] \end{aligned}\right\} \qquad (8)$$

The equations of equilibrium to be satisfied are

$$\frac{\partial T^{11}}{\partial \zeta} + \frac{\zeta}{\zeta^2+\eta^2}\,(3\,T^{11} - T^{22}) = 0 \qquad (9)$$

$$\frac{\partial T^{22}}{\partial \eta} + \frac{\eta}{\zeta^2+\eta^2}\,(3\,T^{22} - T^{11}) = 0 \qquad (10)$$

third being identically satisfied.

On substituting (8) in (9) and (10) and solving, both these equations give the same solution

$$f'^2 + F'^2 = A\,(\zeta^2+\eta^2)^{(2-d)/2} \qquad (11)$$

where $d \equiv \dfrac{2\mu}{\lambda+2\mu}$ and A is an arbitrary constant.

Since f and F are functions of ζ and η respectively, equation (11) will be consistent only if η is small such that η^2 can be neglected and

$$F = B\eta \tag{12}$$

Then, we have

$$f'^2 = -B^2 + D(\zeta)^{2-d} \tag{13}$$

Substituting (12) and (13) in (8), we obtain

$$\left.\begin{aligned} \sigma_{11} &= \frac{\mu}{d}\left[(2-d) + \frac{dB^2}{\zeta^2} - \frac{D}{\zeta^d}\right] \\ \sigma_{22} &= \frac{\mu}{d}\left[(2-d) - \frac{dB^2}{\zeta^2} - \frac{(1-d)\,D}{\zeta^d}\right] \\ \sigma_{33} &= \frac{\mu\,(1-d)}{d}\left[2 - \frac{D}{\zeta^d}\right] \end{aligned}\right\} \tag{14}$$

BOUNDARY CONDITIONS

If the block is bent by applying forces to the edges only, we must have

$$\sigma_{11} = 0 \text{ when } \zeta = \zeta_i \quad i = 1, 2 \tag{15}$$

which gives

$$2 - d + \frac{d\,B^2}{\zeta_i^2} - \frac{D}{(\zeta_i)^d} = 0 \quad i = 1, 2 \tag{16}$$

Solving these equations we obtain

$$B^2 = \frac{2-d}{d} \; \frac{(\zeta_2)^d - (\zeta_1)^d}{(\zeta_2)^{2-d} - (\zeta_1)^{2-d}} \; (\zeta_1\,\zeta_2)^{2-d} \tag{17}$$

$$D = (2-d) \; \frac{(\zeta_2)^2 - (\zeta_1)^2}{(\zeta_2)^{2-d} - (\zeta_1)^{2-d}} \tag{18}$$

On the straight edges $\eta = \pm\,\alpha$, the distribution of tractions gives rise to a Force F_1 and a couple M_1 given by

$$F_1 = \frac{\mu}{d}\left[\frac{2-d}{2}\,(\zeta_2^2 - \zeta_1^2) - dB^2 \log\frac{\zeta_2}{\zeta_1} - \frac{(1-d)\,D}{(2-d)}\,(\zeta_2^{2-d} - \zeta_1^{2-d})\right] \tag{19}$$

$$M_1 = \frac{\mu}{d}\left[\frac{2-d}{4}\,(\zeta_2^4 - \zeta_1^4) - \frac{dB^2}{2}\,(\zeta_2^2 - \zeta_1^2) - \frac{(1-d)\,D}{(4-d)}\,(\zeta_2^{4-d} - \zeta_1^{4-d})\right] \tag{20}$$

The force F_2 required to keep the length of the block constant in the direction of the axis of the shell and the couple M_2 in the axial plane applied to the plane ends of the shell per unit area between η and $\eta + \delta\eta$ are given by

$$F_2 = \frac{\mu\,(1-d)}{d}\left[\frac{2}{3}\,(\zeta_2^3 - \zeta_1^3) - \frac{D}{3-d}\,(\zeta_2^{3-d} - \zeta_1^{3-d})\right] \tag{21}$$

$$-M_2 = \frac{\mu(1-d)}{d}\left[\frac{2}{5}\left(\zeta_2^5-\zeta_1^5\right)-\frac{D}{5-d}\left(\zeta_2^{5-d}-\zeta_1^{5-d}\right)\right] \qquad (22)$$

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