SOME RECURRENCE FORMULAE FOR G-FUNCTION OF TWO VARIABLES-I

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Some recurrence formulae for the G-function of two variables have been obtained after establishing some derivatives.

Agarwal[1] and Sharma[2] defined the G-function of two variables in the form of Mellin-Barnes type integra which has been denoted by Bajpai[3] as

$$G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3; (q_1, q_2), q_3}\left[\begin{array}{c|l} x & (a_{p_1}) ; (c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1}) ; (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] = \frac{1}{(2\pi i)^2} \times$$

$$\times \int_{L_1}\int_{L_2} \frac{\prod_{j=1}^{m_1}\Gamma(b_j - s)\prod_{j=1}^{n_1}\Gamma(1 - a_j + s)\prod_{j=1}^{m_2}\Gamma(d_j - t)\prod_{j=1}^{n_2}\Gamma(1 - c_j + t)\prod_{j=1}^{n_3}\Gamma(1 - e_j + s + t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1 - b_j + s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j - s)\prod_{j=m_2+1}^{q_2}\Gamma(1 - d_j + t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j - t)} \times$$

$$\times \frac{x^s \; y^t}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j - s - t)\prod_{j=1}^{q_3}\Gamma(1 - f_j + s + t)}\, ds\, dt. \qquad (1)$$

The contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops if necessary, to ensure that the poles of $\Gamma(b_j - s), j=1, 2, \ldots\ldots, m_1$ lie on the right and the poles of $\Gamma(1 - a_j + s), j = 1, 2, \ldots\ldots, n_1$ and $\Gamma(1- e_j + s + t), j=1, 2,\ldots\ldots, n_3$ on the left of the contour. Similarly the contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$ with loops if necessary, to ensure that the poles of $\Gamma(d_j - t), j= 1, 2,\ldots\ldots, m_2$ lie on the right and the poles of $\Gamma(1-c_j + t), j=1, 2,\ldots\ldots, n_2$ and $\Gamma(1- e_j + s + t), j=1, 2,\ldots\ldots, n_3$ lie on the left of the contour, provided that

$$0 \leqslant m_1 \leqslant q_1, 0 \leqslant m_2 \leqslant q_2, 0 \leqslant n_1 \leqslant p_1, 0 \leqslant n_2 \leqslant p_2, 0 \leqslant n_3 \leqslant p_3;$$

the integral converges if

$$(p_3 + q_3 + p_1 + q_1) < 2\,(m_1 + n_1 + n_3)\,;\ (p_3 + q_3 + p_2 + q_2) < 2\,(m_2 + n_2 + n_3)\,;$$

$$|\arg x| < [m_1 + n_1 + n_3 - \tfrac{1}{2}\,(p_3 + q_3 + p_1 + q_1)]\,\pi,$$

$$|\arg y| < [m_2 + n_2 + n_3 - \tfrac{1}{2}\,(p_3 + q_3 + p_2 + q_2)]\,\pi.$$

The right hand side of (1) shall, hence-forth be denoted by $G\begin{bmatrix} x^h \\ y^h \end{bmatrix}$, whenever there is no chance of misunderstanding and is the required G-function of two variables. The symbol (a_p) stands for $a_1, a_2, \ldots\ldots\ldots\ldots, a_p$ throughout this paper. Certain recurrence relations for G-function of two variables have already been obtained by the author[4] as particular cases of finite series.

Let us establish the following formulae:

$$x \frac{\partial}{\partial x} G \begin{bmatrix} x^h \\ y^h \end{bmatrix} = hG \left[\begin{array}{c|l} x^h & a_1 - 1, a_2, \ldots, a_{p_1}\,;\,(c_{p_2}) \\ & (e_{p_3}) \\ y^h & (b_{q_1})\ ;\ (d_{q_2}) \\ & (f_{q_3}) \end{array} \right] + h\,(a_1 - 1)\, G \begin{bmatrix} x^h \\ y^h \end{bmatrix} \quad (2)$$

$$x \frac{\partial}{\partial x} G \begin{bmatrix} x^h \\ y^h \end{bmatrix} = h\,(a_{p_1} - 1)\, G \begin{bmatrix} x^h \\ y^h \end{bmatrix} - hG \left[\begin{array}{c|l} x^h & a_1, \ldots, a_{p_1 - 1}, a_{p_1} - 1\,;\,(c_{p_2}) \\ & (e_{p_3}) \\ y^h & (b_{q_1})\,;\,(d_{q_2}) \\ & (f_{q_3}) \end{array} \right] \quad (3)$$

$$x \frac{\partial}{\partial x} G \begin{bmatrix} x^h \\ y^h \end{bmatrix} = hb_1 G \begin{bmatrix} x^h \\ y^h \end{bmatrix} - hG \left[\begin{array}{c|l} x^h & (a_{p_1})\,;\,(c_{p_2}) \\ & (e_{p_3}) \\ y^h & b_1 + 1,\ b_2, \ldots, b_{q_1}\,;\,(d_{q_2}) \\ & (f_{q_3}) \end{array} \right] \quad (4)$$

$$x \frac{\partial}{\partial x} G \begin{bmatrix} x^h \\ y^h \end{bmatrix} = hb_{q_1} G \begin{bmatrix} x^h \\ y^h \end{bmatrix} + hG \left[\begin{array}{c|l} x^h & (a_{p_1});\,(c_{p_2}) \\ & (e_{p_3}) \\ y^h & b_1, \ldots, b_{q_1 - 1}, b_{q_1} + 1\,;\,(d_{q_2}) \\ & (f_{q_3}) \end{array} \right] \quad (5)$$

Similar results hold for $y \frac{\partial}{\partial y} G \begin{bmatrix} x^h \\ y^h \end{bmatrix}$.

Proof : To prove (2), expressing the G-function on the left hand side as Mellin-Barnes type integral (1), changing the order of integration and differentiation, we have

$$h \frac{1}{(2\pi i)^2} \int_{L_1}\int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(b_j - s) \prod_{j=1}^{n_1} \Gamma(1 - a_j + s) \prod_{j=1}^{m_2} \Gamma(d_j - t) \prod_{j=1}^{n_2} \Gamma(1 - c_j + t)}{\prod_{j=m_1+1}^{q_1} \Gamma(1 - b_j + s) \prod_{j=n_1+1}^{p_1} \Gamma(a_j - s) \prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + t) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - t)}$$

$$\times \frac{\prod_{j=1}^{n_3} \Gamma(1 - e_j + s + t)\, s\, x^{hs} y^{ht}}{\prod_{j=n_3+1}^{p_3} \Gamma(e_j - s - t) \prod_{j=1}^{q_3} \Gamma(1 - f_j + s + t)}\, ds\, dt. \tag{6}$$

But $\quad s\,\Gamma(1 - a_1 + s) = [(1 - a_1 + s) + (a_1 - 1)]\, \Gamma(1 - a_1 + s)$

$$= \Gamma(2 - a_1 + s) + (a_1 - 1)\, \Gamma(1 - a_1 + s) \tag{7}$$

Using (7), (6) becomes

$$h \frac{1}{(2\pi i)^2} \int_{L_1}\int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(b_j - s) \prod_{j=2}^{n_1} \Gamma(1 - a_j + s) \prod_{j=1}^{m_2} \Gamma(d_j - t) \prod_{j=1}^{n_2} \Gamma(1 - c_j + t)}{\prod_{j=m_1+1}^{q_1} \Gamma(1 - b_j + s) \prod_{j=n_1+1}^{p_1} \Gamma(a_j - s) \prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + t) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - t)}$$

$$\times \frac{\prod_{j=1}^{n_3} \Gamma(1 - e_j + s + t)\, [\Gamma(2 - a_1 + s) + (a_1 - 1)\, \Gamma(1 - a_1 + s)]\, x^{hs}\, y^{ht}}{\prod_{j=n_3+1}^{p_3} \Gamma(e_j - s - t) \prod_{j=1}^{q_3} \Gamma(1 - f_j + s + t)}\, ds\, dt.$$

Now using (1), the formula (2) is proved.

Proceeding similarly we can prove (3), (4) and (5).

Subtracting (2) from (3), (4) and (5), we get, respectively :

$$(a_{p_1} - a_1)\, G\begin{bmatrix} x^h \\ y^h \end{bmatrix} = G\left[\begin{array}{c|l} x^h & a_1, \ldots, a_{p_1 - 1},\ a_{p_1} - 1 ; (c_{p_2}) \\ & (e_{p_3}) \\ y^h & (b_{q_1}) ; (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] +$$

$$+ G\left[\begin{array}{c|l} x^h & a_1 - 1, a_2, \ldots, a_{p_1} ; (c_{p_2}) \\ & (e_{p_3}) \\ y^h & (b_{q_1}) ; (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] \quad - \qquad (8)$$

A similar result holds for $(c_{p_2} - c_1)\, G\begin{bmatrix} x^h \\ y^h \end{bmatrix}$

$$(b_1 - a_1 + 1)\, G\begin{bmatrix} x^h \\ y^h \end{bmatrix} = G\left[\begin{array}{c|l} x^h & a_1 - 1, a_2, \ldots, a_{p_1} ; (c_{p_2}) \\ & (e_{p_3}) \\ y^h & (b_{q_1}) ; (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] +$$

$$+ G\left[\begin{array}{c|l} x^h & (a_{p_1}) ; (c_{p_2}) \\ & (e_{p_3}) \\ y^h & 1 + b_1, b_2, \ldots, b_{q_1} ; (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] \qquad (9)$$

A similar result holds for $(d_1 - c_1 + 1)\, G\begin{bmatrix} x^h \\ y^h \end{bmatrix}$

$$(b_{q_1} - a_1 + 1)\, G\begin{bmatrix} x^h \\ y^h \end{bmatrix} = G\left[\begin{array}{c|l} x^h & a_1 - 1, a_2, \ldots, a_{p_1} ; (c_{p_2}) \\ & (e_{p_3}) \\ y^h & (b_{q_1}) ; (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] -$$

$$- G\left[\begin{array}{c|l} x^h & (a_{p_1}) ; (c_{p_2}) \\ & (e_{p_3}) \\ y^h & b_1, \ldots, b_{q_1 - 1}, b_{q_1} + 1 ; (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] \qquad (10)$$

A similar result holds for $(d_{q_2} - c_1 + 1)\, G\begin{bmatrix} x^h \\ y^h \end{bmatrix}$

Subtracting (3) from (4) and (5), we get, respectively :

$$(a_{p_1} - b_1 - 1)\ G\begin{bmatrix} x^h \\ y^h \end{bmatrix} = G\left[\begin{array}{c|l} x^h & a_1, \ldots, a_{p_1} - 1, a_{p_1} - 1\ ;\ (c_{p_2}) \\ & (e_{p_3}) \\ y^h & (b_{q_1})\ ;\ (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] -$$

$$- G\left[\begin{array}{c|l} x^h & (a_{p_1})\ ;\ (c_{p_2}) \\ & (e_{p_3}) \\ y^h & 1 + b_1, b_2, \ldots, b_{q_1}\ ;\ (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] \qquad (11)$$

A similar result holds for $(c_{p_2} - d_1 - 1)\ G\begin{bmatrix} x^h \\ y^h \end{bmatrix}$

$$(a_{p_1} - b_{q_1} - 1)\ G\begin{bmatrix} x^h \\ y^h \end{bmatrix} = G\left[\begin{array}{c|l} x^h & a_1, \ldots, a_{p_1} - 1, a_{p_1} - 1\ ;\ (c_{p_2}) \\ & (e_{p_3}) \\ y^h & (b_{q_1})\ ;\ (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] +$$

$$+ G\left[\begin{array}{c|l} x^h & (a_{p_1})\ ;\ (c_{p_2}) \\ & (e_{p_3}) \\ y^h & b_1, \ldots, b_{q_1} - 1, b_{q_1} + 1\ ;\ (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] \qquad (12)$$

A Similar result holds for $(c_{p_2} - d_{q_2} - 1)\ G\begin{bmatrix} x^h \\ y^h \end{bmatrix}$

Subtracting (4) from (5), we get

$$(b_1 - b)\, G\begin{bmatrix} x^h \\ y^h \end{bmatrix} = G\left[\begin{array}{c|l} x^h & (a_{p_1})\ ;\ (c_{p_2}) \\ & (e_{p_3}) \\ y^h & 1 + b_1, b_2, \ldots, b_{q_1}\ ;\ (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] +$$

$$+ G\left[\begin{array}{c|l} x^h & (a_{p_1})\ ;\ (c_{p_2}) \\ & (e_{p_3}) \\ y^h & b_1, \ldots, b_{q_1 - 1}, b_{q_1} + 1\ ;\ (d_{q_2}) \\ & (f_{q_3}) \end{array}\right] \qquad (13)$$

A similar result holds for $(d_1 - d_{q_2})\ G \begin{bmatrix} x^h \\ y^h \end{bmatrix}$.

Because of the symmetry in parameters of the G-function of two variables, the results obtained above can be written in various other forms.

PARTICULAR CASES

Putting $h = 1,\ 1 - a_1 = \gamma_1$; we get the known results[1] from (2) and (9).

Putting $m_2 = q_2 = 1,\ n_2 = n_3 = p_2 = p_3 = q_3 = 0$ and making use of the formula[3], viz.

$$G^{(m,\,1)\,;\,(n,\,0),\,0}_{(p,0),\,0\,;\,(q,\,1),\,0}\left[\begin{array}{c|l} x & (a_p)\,;\ \text{——} \\ & \text{——} \\ y & (b_q)\,;\ 0 \\ & \text{——} \end{array}\right] = e^{-y}\, G^{m,\,n}_{p,\,q}\left[\, x \,\middle|\, \begin{array}{c} (a_p) \\ (b_q) \end{array}\right] \tag{14}$$

we get the known results[5] from (2), (8) and (9).

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