FREE CONVECTION FLOW ALONG AN INFINITE VERTICAL PLATE WITH TIME-VARYING SUCTION

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Free convection flow up a vertical infinite plate subjected to a periodic suction is discussed when the plate temperature oscillates with respect to time over a constant mean. In particular, the response of the mean velocity and temperature profiles to the fluctuating components of the suction velocity and the plate temperature is analysed. For small amplitudes, the mean skin-friction increases with the frequency while the mean heat transfer at the plate is independent of it but varies linearly as the product of the amplitudes. For Prandtl number $P=1$, solutions are also presented when the suction velocity is any arbitrary function of time provided it changes slowly.

Unsteady free convection flow from an infinite vertical plate has been discussed by several authors. Illingworth[1], Rao[2], Nanda & Sharma[3] and Dilip Singh[4] have used the concept of similarity while Menold & Yang[5] and Goldstein & Briggs[6] have applied Laplace transform technique to study this problem for various initial and boundary conditions.

In the present paper, the effect of time-oscillating suction on the free convection past a vertical flat plate is analysed when the plate temperature oscillates in time over a constant mean. A Fourier expansion method is used to obtain an infinite set of coupled equations for the velocity and temperature functions, in which the coupling parameter is the non-dimensional amplitude δ of the unsteady component of suction. Because of this coupling, the mean velocity and temperature distributions are altered by all higher harmonic terms. To solve this set of equations, the functions are expanded in powers of $(\delta \ll 1)$. It is found that the mean profiles are affected by terms of $O(\delta)$. The temperature fluctuations affect only the odd harmonics whereas the fluctuations in the suction velocity affect all the harmonics related to the mean profiles. The mean skin-friction increases with frequency to its asymptotic value of P^{-1}, P being the Prandtl number, but decreases rapidly as the value of P increases. The mean heat transfer upto $O(\delta^2)$ is unaffected by the frequency but depends on the product of the amplitudes of fluctuating components. Solutions are also obtained for the general case when the suction velocity varies slowly with respect to time and is continuously differentiable but otherwise arbitrary.

BASIC EQUATIONS

We consider the flow of a viscous and incompressible fluid along a vertical porous plate of infinite extent. The x-axis is taken along the plate and the y-axis normal to it. Since the plate is infinite all quantities are functions of y and t only. The equations expressing conservation of mass, momentum and energy for the laminar free convection are

$$\frac{\partial v}{\partial y} = 0 \quad , \tag{1}$$

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} = g\,\beta\,(T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2} , \tag{2}$$

$$\frac{\partial v}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial y} , \tag{3}$$

$$\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} , \tag{4}$$

where u, v are the velocity components, T is the fluid temperature and ν, α, g and β respectively denote the kinematic viscosity, thermal diffusivity, acceleration due to gravity and thermal coefficient of volume expansion. In accord with the usual practice in the case of free-convection problems, density has been considered a variable only in forming the buoyancy term $g\,\beta\,(T - T_\infty)$ per unit mass. Viscous dissipation and work done against the gravity field are ignored.

We get from (1

$$v = v\,(t) = - v_0 \left[1 + A\,\delta\,(e^{i\omega t} + e^{-i\omega t}) \right] , \qquad A\,\delta \ll 1 , \tag{5}$$

which represents a steady suction with superimposed weak-fluctuating component. With this, (2) and (4) become

$$\frac{\partial u}{\partial t} - v_0 \left\{ 1 + A\,\delta\,(e^{i\omega t} + e^{-i\omega t}) \right\} \frac{\partial u}{\partial y} = g\,\beta\,(T_\omega - T_\infty)\,\theta + \nu \frac{\partial^2 u}{\partial y^2} , \tag{6}$$

$$\frac{\partial \theta}{\partial t} - v_0 \left\{ 1 + A\,\delta\,(e^{i\omega t} + e^{-i\omega t}) \right\} \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} , \tag{7}$$

where $\theta = (T - T_\infty) \,/\, (T_\omega - T_\infty)$, T_ω being the mean temperature of the plat and T_∞ the ambient temperature. We take the plate temperature to be $T_\omega + B\,(T_\omega - T_\infty)$ $(e^{i\omega t} + e^{-i\omega t})$, which consists of a basic steady distribution with a time-varying quantity $B\,(T_\omega - T_\infty)\,(e^{i\omega t} + e^{-i\omega t})$, B being a positive constant.

The appropriate boundary conditions are

$$\left. \begin{aligned} y = 0 : &\quad u = 0 , \quad \theta = 1 + B\,(e^{i\omega t} + e^{-i\omega t}), \\ y \to \infty : &\quad u \to 0 , \quad \theta \to 0 \end{aligned} \right\} \tag{8}$$

The method of solution is similar to the one adopted by Kelly[7], who has investigated the flow over a plate with periodic suction.

FOURIER SERIES SOLUTIONS AND RESULTS

Equation (3) gives the pressure distribution as

$$p - p_0 = \rho y\, v_0 \frac{d}{dt} \left(e^{i\omega t} + e^{-i\omega t} \right) , \tag{9}$$

p_0 being the pressure at the plate.

To solve (6) to (8), following Kelly[7], we assume solutions in the form

$$u(y, t) = u_0(y) + \sum_{n=1}^{\infty} u_n(y)\, e^{in\omega t} + \sum_{n=1}^{\infty} \bar{u}_n(y)\, e^{-in\omega t}, \tag{10}$$

$$\theta(y, t) = \theta_0(y) + \sum_{n=1}^{\infty} \theta_n(y)\, e^{in\omega t} + \sum_{n=1}^{\infty} \overline{\theta_n}(y)\, e^{-in\omega t}. \tag{11}$$

Substituting the above series into (6) and (7) and equating terms of powers of $e^{i\omega t}$ and $e^{-i\omega t}$, we get in the non-dimensional form

$$\frac{d^2\phi_0}{d\eta^2} + \frac{d\phi_0}{d\eta} + \delta A\left(\frac{d\phi_1}{d\eta} + \frac{d\bar{\phi}_1}{d\eta}\right) + \theta_0 = 0, \tag{12}$$

$$\frac{1}{P}\frac{d^2\theta_0}{d\eta^2} + \frac{d\theta_0}{d\eta} + \delta A\left(\frac{d\theta_1}{d\eta} + \frac{d\bar{\theta}_1}{d\eta}\right) = 0, \tag{13}$$

$$\frac{d^2\phi_n}{d\eta^2} + \frac{d\phi_n}{d\eta} - in\lambda\,\phi_n + \delta A\left(\frac{d\phi_{n-1}}{d\eta} + \frac{d\phi_{n+1}}{d\eta}\right) + \theta_n = 0, \quad n \geq 1, \tag{14}$$

$$\frac{1}{P}\frac{d^2\theta_n}{d\eta^2} + \frac{d\theta_n}{d\eta} - in\lambda\,\theta_n + \delta A\left(\frac{d\theta_{n-1}}{d\eta} + \frac{d\theta_{n+1}}{d\eta}\right) = 0, \quad n \geq 1, \tag{15}$$

where $\eta = \dfrac{v_0 y}{\nu}, \quad \phi_n = \dfrac{u_n}{U}$

U being a characteristic velocity such that

$\nu g \beta (T_\omega - T_\infty) / U v_0^2 = 1$

$P = \nu/\alpha$ is the Prandtl number

and $\lambda = \nu\omega/v_0^2$ is the frequency parameter.

The boundary conditions become

$$\left.\begin{aligned} &\phi_j(0) = \phi_j(\infty) = 0, \qquad j \geq 0, \\ &\theta_0(0) = 1, \quad \theta_0(\infty) = 0; \ \theta_1(\infty) = B, \ \bar{\theta}_1(\infty) = 0; \\ &\theta_k(0) = \theta_k(\infty) = 0, \quad k \geq 2 \end{aligned}\right\} \tag{16}$$

From (12) to (15) we find that the mean flow and the mean temperature are affected by the oscillations through ϕ_1 and θ_1 and their conjugates and consequently through all the higher harmonic terms. Such distortion is usually associated with non-linear type of equations. Here the distortion occurs in a linear problem and is due to the time-dependent suction which causes the equation to have a variable coefficient. Distortion of this type would not occur if the suction were constant.

In order to solve the infinite set of coupled differential equations, δ must be restricted in some manner. For $\delta \ll 1$, the equations are weakly coupled and following Kelly[7], we expand ϕ_n and θ_n as

$$\phi_n(\eta) = \sum_{j=0}^{\infty} \phi_{nj}(\eta)\ \delta^j, \quad n \geqslant 0, \tag{17}$$

$$\theta_n(\eta) = \sum_{j=0}^{\infty} \theta_{nj}(\eta)\ \delta^j, \quad n \geqslant 0. \tag{18}$$

Substituting the above series into (12) to (15) and comparing coefficients of like powers of δ, we obtain the following set of coupled equations for various ϕ and θ.

$$\left.\begin{aligned} \phi_{00}'' + \theta_{00}' &= -\theta_{00} , \\ \theta_{00}'' + P\,\theta'_{00} &= 0 \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} \phi_{10}'' + \phi_{10}' - i\lambda\,\phi_{10} &= -\theta_{10}, \\ \theta_{10}'' + P\,\theta_{10}' - i\lambda P\,\theta_{10} &= 0 , \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} \phi_{20}'' + \phi_{20}' - 2i\lambda\,\phi_{20} &= -\theta_{20} , \\ \theta_{20}'' + P\,\theta_{20}' - 2i\lambda\,\theta_{20} &= 0 , \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} \phi_{01}'' + \phi_{01}' &= -A\,(\phi_{10}' + \bar{\phi}'_{10}) - \theta_{01} , \\ \theta_{01}'' + P\,\theta_{01}' &= -P A\,(\theta_{10}' + \bar{\theta}'_{10}) , \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} \phi_{11}'' + \phi_{11}' - i\lambda\,\phi_{11} &= -A\,(\phi_{00}' + \phi_{20}') - \theta_{11} , \\ \theta_{11}'' + P\,\theta_{11}' - i\lambda\,\theta_{11} &= -P A\,(\theta_{00}' + \theta_{20}') , \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} \phi_{02}'' + \phi_{02}' &= -A\,(\phi_{11}' + \bar{\phi}'_{11}) - \theta_{02} , \\ \theta_{02}'' + P\,\theta_{02} &= -A P\,(\theta_{11}' + \bar{\theta}'_{11}) , \end{aligned}\right\} \tag{24}$$

with the boundary conditions

$$\left.\begin{aligned} &\phi_{rj}(0) = \phi_{rj}(\infty) = 0 , \quad r \geqslant 0 , j \geqslant 0 , \\ &\theta_{00}(0) = 1 , \ \theta_{00}(\infty) = 0 ; \ \theta_{10}(0) = B, \ \theta_{10}(\infty) = 0 ; \\ &\theta_{kl}(0) = \theta_{kl}(\infty) = 0 , \quad k \geqslant 2 , \ l \geqslant 0 \end{aligned}\right\} \tag{25}$$

The dashes now denote differentiation with respect to η.

Equations (19) to (24) have been solved subject to (25) but in order to conserve space, they are not presented here. The solutions for the mean profiles to $O(\delta^2)$ may be expressed as

$$\frac{u_0\ (\lambda, \eta)}{U} = \phi_{00}(\lambda, \eta) + \delta\ \phi_{01}(\lambda, \eta) + \delta^2\ \phi_{02}(\lambda, \eta) \tag{26}$$

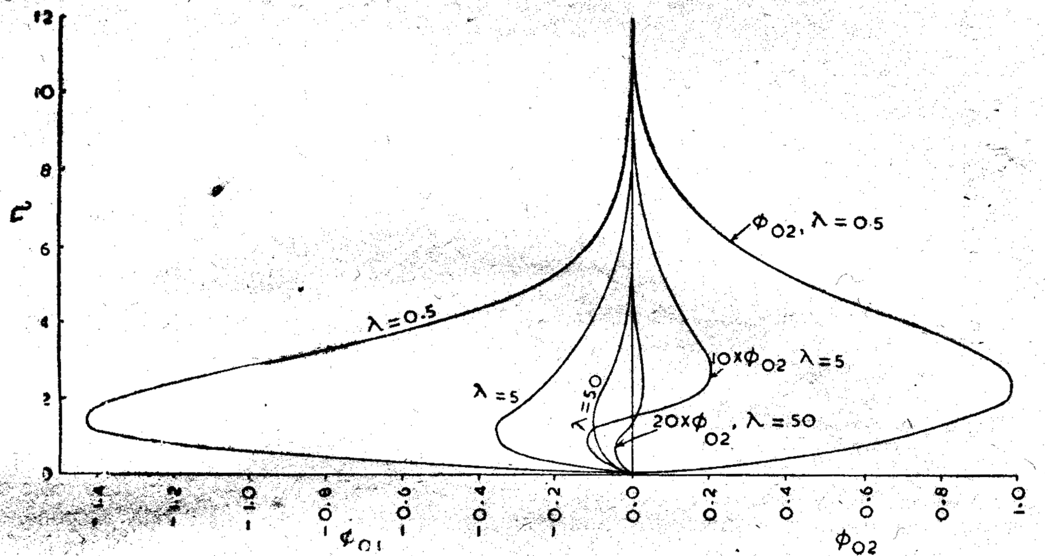

Fig. 1—Functions which modify the mean velocity profile, (P=0·72).

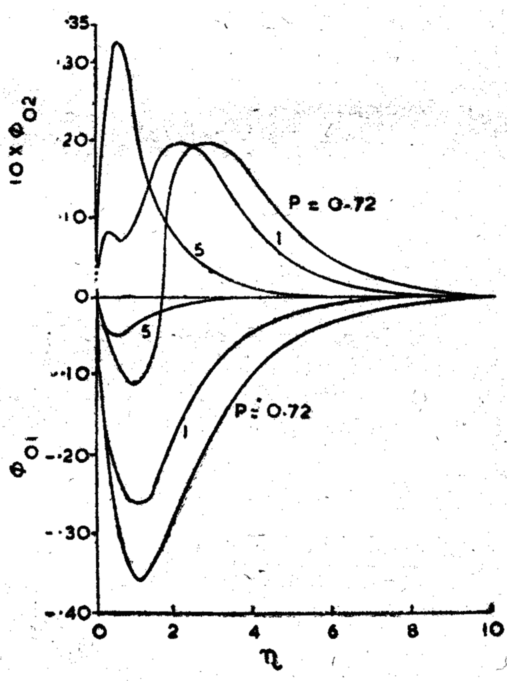

Fig. 2—Functions which modify the mean velocity profile. (λ=5).

and

$$\theta_0(\lambda, \eta) = \theta_{00}(\lambda, \eta) + \delta\theta_{01}(\lambda, \eta) + +\delta^2\ \theta_{02}(\lambda, \eta) \quad (27)$$

It is easily seen from the differential equations and the boundary conditions that if the plate was to be kept at a constant temperature (*i.e.*, $B = 0$), $\phi_{01} = \theta_{01} = 0$ and presumably, $\phi_{oj} = \theta_{oj} = 0$, j being odd. It follows that the functions ϕ_{ok} and θ_{ok}, k being even, are unaffected by the temperature fluctuations of the plate.

The velocity and temperature functions which affect the mean flow and temperature are displayed in Fig. 1 to 4 for $A = B = 1$, $\lambda = 0{\cdot}5$, 5, 50 and $P = 0{\cdot}72$, 1, 5. From Fig. 1 and 3 we find that ϕ_{02} and θ_{02} are of the same order as ϕ_{01} and θ_{01} respectively for smaller frequencies and of lesser order for higher frequencies. This suggests rapid convergence of the series for ϕ_0 and θ_0 for higher frequencies and that their

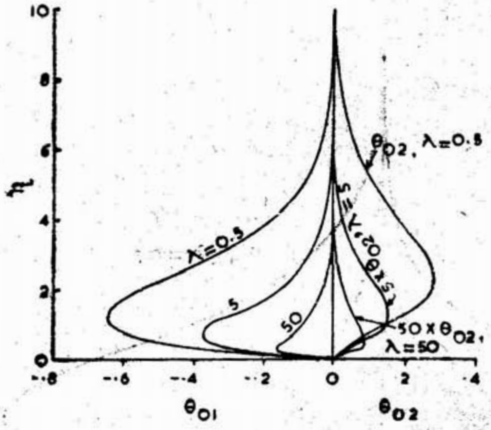

Fig. 3—Functions which modify the mean temperature distribution, (P=0·72)

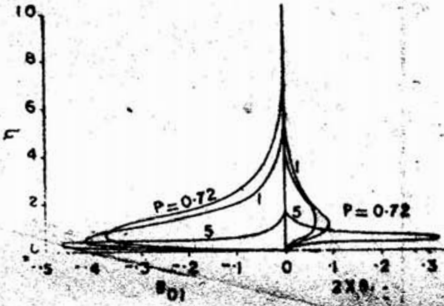

Fig. 4—Functions which modify the mean temperature distribution, (λ=5).

expansions as given by (17) and (18) may not be valid for $\delta > 1$ in the case of low frequencies. The mean flow and temperature nearer to the wall only respond to higher frequencies. The Prandtl number has the effect of increasing the disturbance in the mean temperature of the fluid and at the same time suppressing the region of disturbance closer to the wall.

The mean skin-friction on the plate is given by

$$\tau_0 = \rho\nu \left(\frac{\partial u_0}{\partial y} \right)_{y=0} = \rho\, U\, v_0 \left(\frac{d\phi_0}{d\eta} \right)_{\eta=0}$$

In the non-dimensional form it is obtained as

$$\frac{\tau_0}{\rho U\, v_0} = \frac{1}{P} + \frac{2A}{h_1^2 + h_2^2} \left\{ - Bh_1\delta + \frac{A\,P\,h_2}{\lambda}\,\delta^2 \right\} + O\,(\delta^3), \qquad (28)$$

where

$$h_1 + i\,h_2 = \frac{P}{2} \left[1 + \left(1 + \frac{4\,\lambda i}{P} \right)^{\frac{1}{2}} \right]$$

Table 1 gives the values of the skin-friction coefficient for $\delta = 0{\cdot}1,\ 0{\cdot}3,\ 0{\cdot}5$ and $A = B = 1$. We find that the dimensionless coefficient of skin-friction increases with frequency to its asymptotic value P^{-1} but decreases rapidly as P increases. Also we note that the skin-friction coefficient decreases as the amplitude of the fluctuating component of suction velocity increases.

The local mean heat transfer at the plate is given by

$$q = -K \left(\frac{\partial T}{\partial y} \right)_{y=0} = -K\, \frac{(T_\omega - T_\infty)\, v_0}{\nu} \left(\frac{d\theta}{d\eta} \right)_{\eta=0}$$

or

$$\frac{\nu q}{K\,(T_\omega - T_\infty)\, v_0} = P\,(1 + 2\,A\,B\,\delta) + O(\delta^3), \qquad (29)$$

TABLE 1

DIMENSIONLESS COEFFICIENT OF THE MEAN SKIN-FRICTION

δ	λ	P 0·2	0·72	1·0	5·0	10
0·1	0·5	4·5854	1·2030	0·8536	0·1646	0·0821
	5·0	4·8598	1·3165	0·9392	0·1762	0·0854
	50·0	4·9553	1·3654	0·9801	0·1912	0·0939
0·3	0·5	3·8209	0·9012	0·6262	0·1166	0·0581
	5·0	4·5822	1·1765	0·8230	0·1358	0·0626
	50·0	4·8660	1·3186	0·9405	0·1741	0·0823
0·5	0·5	3·1429	0·6930	0·4859	0·0991	0·0499
	5·0	4·3086	1·0430	0·7140	0·1050	0·0486
	50·0	4·7769	1·2720	0·9011	0·1576	0·0714
∞		5·0000	1·3889	1·0000	0·2000	0·1000

K being the thermal conductivity. Hence the heat transfer upto $O(\delta^2)$ is independent of frequency and varies linearly as the product of the amplitudes.

ARBITRARY SUCTION VELOCITY

We assume now that the suction velocity $-v(t)$ $[v(t) > 0]$ varies slowly with respect to time but is otherwise arbitrary. Also let the plate be maintained at a constant temperature. We take

$$u(y, t) = U \phi(\eta, t), \tag{30}$$

where $\eta = \dfrac{v(t)\, y}{\nu}$ and U is a characteristic velocity such that $\nu g \beta (T_\omega - T_\infty)/U\ v^2 = 1$

Substituting (30) into (2) and (4), we get

$$\frac{\nu}{v^2}\frac{\partial \phi}{\partial t} + \frac{\nu \dot{v}}{v^3}\eta\frac{\partial \phi}{\partial \eta} - \frac{\partial \phi}{\partial \eta} = \theta + \frac{\partial^2 \phi}{\partial \eta^2}, \tag{31}$$

$$\frac{\nu}{v^2}\frac{\partial \theta}{\partial t} + \frac{\nu \dot{v}}{v^3}\eta\frac{\partial \theta}{\partial \eta} - \frac{\partial \theta}{\partial \eta} = \frac{1}{P}\frac{\partial^2 \theta}{\partial \eta^2}, \tag{32}$$

where the dot denotes differentiation with respect to t, and $\theta = (T - T_\infty)/(T_\omega - T_\infty)$ Following Kelly[7], we expand ϕ as

$$\phi(\eta, t) = \phi_0(\eta) + \frac{\nu \dot{v}}{v^3}\phi_1(\eta) + \frac{\nu^2 \ddot{v}}{v^5}\phi_2(\eta) + \frac{\nu^2 \dot{v}^2}{v^6}\phi_3(\eta) + \ldots\ldots \tag{33}$$

and similarly in the case of $\theta(\eta, t)$. Substituting these two series into equations (31) and (32), and comparing like terms, we realise a set of linear differential equations for var ous ϕ_j and θ_j. Their solutions subject to

$$\left.\begin{aligned} &\phi_j(0) = \phi_j(\infty) = 0 \quad, \ j \geqslant 0, \\ &\theta_0(0) = 1, \ \theta_0(\infty) = 0, \ \theta_k(0) = \theta_k(\infty) = 0, \ k \geqslant 1 \end{aligned}\right\} \tag{34}$$

are as follows :

$$\left.\begin{aligned}\phi_0(\eta) &= \eta\, e^{-\eta} \;, \\ \theta_0(\eta) &= e^{-\eta} \;,\end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned}\phi_1(\eta) &= \tfrac{1}{2}\,(6\,\eta + 3\,\eta^2 + \eta^3)\, e^{-\eta} \;, \\ \theta_1(\eta) &= \tfrac{1}{2}\,(2\,\eta + \eta^2)\, e^{-\eta} \;,\end{aligned}\right\} \tag{36}$$

$$\left.\begin{aligned}\phi_2(\eta) &= -\tfrac{1}{6}\,(84\,\eta + 42\eta^2 + 9\eta^3 + \eta^4)\, e^{-\eta} \;, \\ \theta_2(\eta) &= -\tfrac{1}{6}\,(12\eta + 6\eta^2 + \eta^3)\, e^{-\eta}\end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned}\phi_3(\eta) &= \tfrac{1}{24}(1464\,\eta + 732\,\eta^2 + 188\,\eta^3 + 30\eta^4 + 3\eta^5)\, e^{-\eta} \;, \\ \theta_3(\eta) &= \tfrac{1}{8}\,(64\eta + 32\eta^2 + 8\eta^3 + \eta^4)\, e^{-\eta} \;,\end{aligned}\right\} \tag{38}$$

The wall skin-friction in this case is given by

$$\tau = \rho\, U\, v \left[\, 1 + \frac{3\,\nu\,\dot{v}}{v^3} + \frac{14\,\nu^2\,\ddot{v}}{v^5} + \frac{61\,\nu^2\,\dot{v}^2}{v^6} + \ldots \right] \tag{39}$$

and the heat-transfer rate at the plate is obtained as

$$q = \frac{K\, v\,(T_\omega - T_\infty)}{\nu} \left[\, 1 + \frac{\nu\,\dot{v}}{v^3} + \frac{2\,\nu^2\,\ddot{v}}{v^5} - \frac{8\,\nu^2\,\dot{v}^2}{v^6} + \ldots \right] \tag{40}$$

Thus, a suction velocity which increases with time, increases the skin-friction but decreases the heat transfer compared to their quasi-steady values.

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