GEOMETRY OF THE VORTEXLINE IN COMPLEX-LAMELLAR STEADY DIABATIC GAS FLOWS

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Considering the geometry of triply orthogonal spatial curves of congruences in Euclidean space E^3, formed by a vortexline and its principal normal and binormal, various kinematic and kinetic properties of an inviscid diabatic steady gas flow are examined. Extending the same technique complex-lamellar flows are studied.

The non-linear character of differential equations governing fluid dynamic problems has presented considerable difficulties to find out exact possible flows. Consequently many interesting devices (inverse, semi-inverse and superposability) have been introduced. The introduction of the Geometric theory of surfaces and curves in fluid flow theory is also one such device, which presents the flow possible. In this paper we discuss the geometric properties of steady diabatic complex-lamellar gas flows, correlating the geometry of vortexline. It is assumed that the fluid is non-viscous and subject to no extraneous forces. Complex-lamellar flows are characterized by the fact that there exists a family of ∞ of surfaces orthogonal to the streamlines[1]. Each surface is a Beltrami surface.

Defining complex-lamellar velocity field it is observed that the streamlines and the vortexlines intersect orthogonally and each Beltrami surface contains the vortexline. From the geometry of Beltrami surface it is seen that the velocity vector field lies in the normal plane to a vortexline. Further the velocity shall be parallel to the normal to a Beltrami surface if the vortexlines are geodesics. The necessary geometric conditions to be satisfied by the vortexline geometric parameters are established and we observe that the resolved parts of velocity along a principal normal and binormal to a vortexline cannot be uniform simultaneously. The magnitude of the vorticity shall be uniform along a vortexline if the normal congruences are minimal and the converse is also true. Transforming the equation of continuity, the curvature distribution of a vortexline is studied. Decomposing the momentum equation it is proved that the total pressure remains uniform along a vortexline. The normals to the isobars cannot be parallel to the streamlines, whether the fluid is adiabatic or diabatic. The basic integrability conditions for a diabatic complex-lamellar steady flows are obtained in geometric parameters of the vortexline.

The adiabatic phenomena can be discussed as a special case of this investigation.

BASIC EQUATIONS

The basic equations governing steady diabatic, inviscid gas flow in the absence of extraneous forces, in Crocco's velocity vector field are given below in the usual notation[2] :

$$div\left\{ \overrightarrow{W}(1-W^2)^{\frac{1}{\gamma-1}} \right\} = q\left(1+\frac{\gamma+1}{\gamma-1}.W^2\right)(1-W^2)^{\frac{2-\gamma}{\gamma-1}} \tag{1}$$

$$\nabla \log p_t = \left(\frac{\overrightarrow{W} \wedge \text{Curl}\, \overrightarrow{W} - q\,\overrightarrow{W}}{1-W^2}\right)\frac{2\gamma}{\gamma-1} \tag{2}$$

$$T_t = T\,(1-W^2)^{-1} \tag{3}$$

$$p_t = p\,(1-W^2)^{\frac{-\gamma}{-1}} \tag{4}$$

In addition to these we add the Crocco's vorticity equation for adiabatic gas flow :

$$V_t^2\, \overrightarrow{W} \wedge \text{Curl}\, \overrightarrow{W} + W^2\, V_t\, \nabla V_t = C_p\, \nabla T_t - T\nabla S \tag{5}$$

where $\overrightarrow{W}$, q, γ, p_t, V_t, C_p, T_t, T, S, and W are reduced velocity vector, the heat content, the adiabatic exponent, the total pressure, the limiting velocity, the stagnation enthalpy, the temperature, the specific entropy and the magnitude of the Crocco's velocity vector field respectively.

COMPLEX-LAMELLAR FLOWS

Complex-lamellar flows are characterized by the fact that there exists a family of ∞ surfaces[2] orthogonal to the streamlines [1,3] and each surface is a Beltrami surface.

The velocity vector of this field can be expressed as :

$$\overrightarrow{W} = \alpha \text{ grad } \phi \tag{6}$$

where α and ϕ are scalar point functions and $\phi\,(\vec{r})$ = constant surface of the Beltrami surface. For this field the condition for the streamlines to be intersected normally by a one parameter family of surfaces, leads

$$\overrightarrow{W}.\ \text{Curl}\ \overrightarrow{W} = 0 \quad \text{or} \quad \nabla\phi.\ \text{Curl}\ \overrightarrow{W} = 0 \tag{7}$$

This shows that the streamlines and vortexlines intersect orthogonally and the Beltrami surfaces contain the vortexlines for a complex-lamellar flow.

Using the geometry of the vortexlines established earlier[4], the velocity vector $\overrightarrow{W}$ can be expressed as

$$\vec{W} = \vec{n}\, W_n + \vec{b}\, W_b \tag{8}$$

where W_n and W_b are the components of the velocity along a principal normal and a binormal to a vortexline.

Operating Curl on (8) and equating to ($\vec{t}\ \zeta$) where ζ is the magnitude of the vorticity and $\vec{t}$ is the unit tangent vector to a vortexline, we obtain

$$\zeta = \frac{dW_b}{dn} - \frac{dW_n}{db} \tag{9}$$

$$W_n\,(\tau + \sigma'') - K'' W_b + \frac{dW_b}{ds} = 0 \tag{10}$$

$$\frac{dW_n}{ds} - KW_n + (\sigma' - \tau)\,W_b = 0 \tag{11}$$

where $\left(\frac{d}{ds}, \frac{d}{dn}, \frac{d}{db}\right)$, (k, k', k'') and $(\tau, \sigma', \sigma'')$ are the directional derivatives, the curvatures, and the torsions of the vortexlines, the principal normals and their binormals respectively.

Equations (9) to (11) constitute the basic conditions to be satisfied by a complex-lamellar velocity vector field, in the language of vortexline geometry. Also from (9) we observe that the components of the velocity along a binormal and principal normal cannot be uniform simultaneously along $\vec{n}$ and $\vec{b}$ respectively.

Now making use of solenoidal property of the vorticity together with (9) to (11), we obtain

$$W_n \left\{ \frac{dk}{db} - k\,(\tau + \sigma'') - \frac{d}{dn}(\tau + \sigma'') \right\} + (k'' - k')\frac{dW_b}{db}$$
$$+ (k + k' + k'')\frac{dW_n}{db} + W_b \left\{ k' k'' + \frac{dk''}{dn} - \frac{d}{db}(\sigma' - \tau) \right\}$$
$$+ (k'' - \sigma' - \sigma'' - 2\tau)\frac{dW_n}{dn} - (k' + k'')\frac{dW_b}{dn}$$
$$+ \frac{d}{ds}\left(\frac{dW_n}{dn} - \frac{dW_n}{db}\right) = 0 \tag{12}$$

This can also be written as

$$k' + k'' = \frac{d}{ds} \log \zeta \tag{13}$$

From this it is evident that the normal congruences are minimal if the magnitude of the vorticity is uniform along an individual vortexline, and the converse is also true.

Making use of (8) in (1), we can decompose the continuity equation into intrinsic form as

$$kW_n = \frac{dW_n}{d_n} + \frac{dW_b}{d_b} + \left(W_n \frac{d}{d_n} + W_b \frac{d}{d_b} \right) \log (1 - W^2)^{\frac{1}{\gamma - 1}}$$
$$- \frac{q}{1 - W^2} \left(1 + \frac{\gamma + 1}{\gamma - 1} W^2 \right) \tag{14}$$

This expresses the curvature distribution of a vortexline flow, for a complex-lamellar velocity field. If the vortexline is geodesic on a Beltrami surface (14) simplifies to

$$k = \frac{d}{dn} \left\{ \log W (1 - W^2)^{\frac{1}{\gamma - 1}} \right\} - \frac{q}{1 - W^2} \left(1 + \frac{\gamma + 1}{\gamma - 1} W^2 \right) \tag{15}$$

From this we observe that the vortexlines are straight if the flow is adiabatic and velocity uniform along the principal normal to a vortexline.

Forming the scalar products of (2) by $\vec{t}$, $\vec{n}$ and $\vec{b}$ successively we obtain the following

$$\frac{dp_t}{ds} = 0 \tag{16}$$

$$\frac{dp_t}{dn} = \frac{2\gamma p_t}{(\gamma - 1)(1 - W^2)} (\zeta W_b - q W_n) \tag{17}$$

$$\frac{dp_t}{db} = \frac{-2\gamma p_t}{(\gamma - 1)(1 - W^2)} (\zeta W_n + q W_b) \tag{18}$$

From (16) we see that the total pressure remains uniform along a vortexline. Also eliminating ζ from (17) and (18) we obtain :

$$\vec{W} \cdot \nabla p_t = \frac{-2\gamma q p_t W^2}{(\gamma - 1)(1 - W^2)} \tag{19}$$

We conclude from (18) and (19) that the streamlines and the vortexlines intersect orthogonally along an individual isobars if the flow is adiabatic.

Also eliminating q the heat content from (14) and (18) we obtain

$$\nabla p_t \wedge \vec{W} = \frac{2 p_t \gamma W^2 \vec{\zeta}}{(\gamma - 1)(1 - W^2)} \tag{20}$$

It is clear that the normals to isobar and the streamlines cannot be parallel for *rotational* diabatic or adiabatic steady gas flow.

Also from (19) and (20) we obtain

$$n\pi - \theta = \tan^{-1} (\zeta / q) \tag{21}$$

where $n = 0, 1, 2, \ldots,$ and θ is the inclination of the streamline with a normal to an isobar

Using (4) in (2) and decomposing into intrinsic form we obtain :

$$\frac{dp}{ds}=\frac{-2\gamma\, pW\dfrac{dW}{ds}}{(\gamma-1)(1-W^2)} \tag{22}$$

$$\frac{1}{p}\frac{dp}{dn}=\frac{-2\gamma}{(\gamma-1)(1-W^2)}\left\{W\frac{dW}{dn}+q\,W_n-\zeta W_b\right\} \tag{23}$$

$$\frac{1}{p}\frac{dp}{db}=\frac{-2\gamma}{(\gamma-1)(1-W^2)}\left(W\frac{dW}{db}+\zeta\,W_n+qW_b\right) \tag{24}$$

These give the variations of pressure along a vortexline, principal normal and binormal. The adiabatic case can be discussed as a special case.

Operating Curl on (2) and using (8) we decompose as

$$\frac{d}{db}\left(\frac{\zeta W_b-qW_n}{1-W^2}\right)+\frac{d}{dn}\left(\frac{\zeta W_n+qW_b}{1-W^2}\right)=0 \tag{25}$$

$$\frac{(\tau+\sigma'')(\zeta W_b-qW_n)}{1-W^2}+\frac{k''(\zeta W_n+qW_b)}{1-W^2}-\frac{d}{ds}\left(\frac{\zeta W_n+qW_b}{1-W^2}\right)=0 \tag{26}$$

$$\frac{d}{ds}\left(\frac{\zeta W_b-qW_n}{1-W^2}\right)-k'\left(\frac{\zeta W_b-qW_n}{1-W^2}\right)-(\sigma'-\tau)\left(\frac{\zeta W_n+qW_b}{1-W^2}\right)=0 \tag{27}$$

These constitute the basic integrability conditions for diabatic complex-lamellar steady gas flow in intrinsic form.

Using (3) in (5) we obtain

$$\frac{dS}{ds}=\frac{C_p}{1-W^2}\left(\frac{d}{ds}\log T+\frac{2W}{1-W^2}\frac{dW}{ds}\right)-\frac{W^2V_t}{T}\frac{dV_t}{ds} \tag{28}$$

$$\frac{dS}{dn}=\frac{C_p}{1-W^2}\left(\frac{d}{dn}\log T+\frac{2W}{1-W^2}\frac{dW}{dn}\right)-\frac{W^2V_t}{T}\frac{dV_t}{dn}-\frac{V_t^2W_b\zeta}{T} \tag{29}$$

$$\frac{dS}{db}=\frac{C_p}{1-W^2}\left(\frac{d}{db}\log T+\frac{2W}{1-W^2}\frac{dW}{db}\right)-\frac{W^2V_t}{T}\frac{dV_t}{db}+\frac{V_t^2\zeta W_n}{T} \tag{30}$$

These give the variation of the specific entropy along a vortexline and its principal normal and binormal.

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