FOURIER SERIES FOR FOX'S *H*-FUNCTION

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Two integrals involving Fox's H-function have been evaluated and used to establish two Fourier series for the H-function. On specialising the parameters, the H-function can be reduced to Meijer G-function, Mac Robert's E-function, generalised hypergeometric functions and many other higher transcedental functions[1]. The results established are of a general character.

Carlson & Greiman[2] have obtained a cosine series for Gegenbauer's function. MacRobert[3,4] has established a cosine and sine series for MacRobert's E-function. Jain[5], Kesarwani[6] and Bajpai[7-9] have obtained some Fourier series for Meijer's G-function. Except a few[7,9] all the Fourier series for the G-function have been established following the results of MacRobert[3,4]. The Fourier series for H-function in this paper have been obtained with the help of a result given by Nielsen[10].

The H-function introduced by Fox[11] is represented and defined as follows :

$$H_{p,\,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,\,e_1),\,\ldots\ldots\ldots,\,(a_p,\,e_p)\\(b_1,\,f_1),\,\ldots\ldots\ldots,\,(b_q,\,f_q)\end{matrix}\right]$$

$$=\frac{1}{2\,\pi\,i}\int_L\frac{\prod\limits_{j=1}^{m}\Gamma\,(b_j-f_j\,s)\,\prod\limits_{j=1}^{n}\Gamma\,(1-a_j+e_j\,s)\,z^s\,ds,}{\prod\limits_{j=m+1}^{q}\Gamma\,(1-b_j+f_j\,s)\prod\limits_{j=n+1}^{p}\Gamma\,(a_j-e_j\,s)}\,, \tag{1}$$

where an empty product is interpreted as 1, $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$; e's and f's are all positive numbers, L is suitable contour of Barnes type such that the poles of $\Gamma\,(b_j-f_j\,s), j=1,\ldots\ldots,$ m lie on the right hand side of the contour and those of $\Gamma\,(1-a_j+e_j\,s)$, $j=1,\ldots\ldots\ldots, n$ lie on the left hand side of the contour.

Asymptotic expansion and analytic continuation of the H-function have been discussed by Braaksma[12].

Following formulae are required in the proofs :

(*a*) The following integrals[10].

$$\int_0^{\pi}(\sin\,\theta)^{\rho}\,\cos u\theta\,d\theta=\frac{\pi\,\Gamma\,(1+\rho)\,\cos\,(\pi\,u/2)}{2^{\,\delta}\,\Gamma\left(1+\dfrac{\rho+u}{2}\right)\Gamma\left(1+\dfrac{\rho-u}{2}\right)}\,, \tag{2}$$

$$\int_0^{\pi} (\sin\,\theta)^{\rho} \sin u\,\theta\,d\,\theta = \frac{\pi\,\Gamma\,(1+\rho)\,\sin\,(\pi\,u/2)}{2\rho\Gamma\left(1+\frac{\rho+u}{2}\right)\Gamma\left(1+\frac{\rho-u}{2}\right)}, \qquad (3)$$

where $\rho > -1$

(*b*) The duplication formula for the gamma-function[13].

$$(\pi)^{\frac{1}{2}}\,\Gamma\,(2\,z) = 2^{(2Z-1)}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}). \qquad (4)$$

In what follows for sake of brevity (a_p, e_p) denotes $(a_1, e_1), \ldots\ldots, (a_p, e_p)$; a_p stand for $a_1, \ldots\ldots, a_p$ and the symbol $\triangle\,(\delta, \alpha)$ represents the set of parameters $\alpha/\delta,\ \frac{\alpha+1}{\delta}, \ldots\ldots, \frac{\alpha+\delta-1}{\delta}$, where δ is a positive integer.

THE INTEGRALS

The integrals to be established are :

$$\int_0^{\pi} (\sin\,\theta)^{\rho}\,\cos u\theta\; H_{p,\,q}^{m,\,n}\left[z\,(\sin\,\theta)^{2\delta} \left| \begin{matrix} (a_p,\,e_p) \\ (b_q,\,f_q) \end{matrix} \right.\right] d\theta$$

$$= (\pi)^{\frac{1}{2}} \cos\frac{\pi\,u}{2}\; H_{p+2,\,q+2}^{m,\,n+2}\left[z \left| \begin{matrix} \left(\frac{1-\rho}{2},\,\delta\right), (-\rho/2,\,\delta),\,(a_p,\,e_p) \\ (b_q, f_q), \left(-\frac{\rho+u}{2},\,\delta\right), \left(-\frac{\rho-u}{2},\,\delta\right) \end{matrix} \right.\right] \qquad (5)$$

$$\int_0^{\pi} (\sin\,\theta)^{\rho}\,\sin u\theta\; H_{p,\,q}^{m,\,n}\left[z\,(\sin\,\theta)^{2\delta} \left| \begin{matrix} (a_p,\,e_p) \\ (b_q,\,f_q) \end{matrix} \right.\right] d\theta$$

$$= (\pi)^{\frac{1}{2}} \sin\frac{\pi\,u}{2}\; H_{p+2,\,q+2}^{m,\,n+2}\left[z \left| \begin{matrix} \left(\frac{1-\rho}{2},\,\delta\right), (-\rho/2,\,\delta),\,(a_p,\,e_p) \\ (b_q, f_q), \left(-\frac{\rho+u}{2},\,\delta\right), \left(-\frac{\rho-u}{2},\,\delta\right) \end{matrix} \right.\right] \qquad (6)$$

where δ is a positive number and

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \leqslant 0, \quad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv K > 0,$$

$|\arg\,z| < \frac{1}{2}\,K\,\pi$, $Re\,2\,\delta\,b_j/f_j > 1-\rho \quad (j = 1, \ldots\ldots, m)$.

Proof

To prove (5), expressing the H-function in the integrand as a Mellin-Barnes type integral (1) and interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)\,z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)}\int_0^{\pi}(\sin\theta)^{\rho+2\delta s}\cos u\theta\,d\theta\,ds.$$

Now evaluating the inner integral with the help of (2) and using duplication formula for Gamma-function (4), we get

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)\sqrt{\pi}\,\Gamma\left(\frac{1+\rho}{2}+\delta s\right)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)\,\Gamma\left(1+\frac{\rho+u}{2}+\delta s\right)}\times$$

$$\times\frac{\Gamma(1+\rho/2+\delta s)\cos(\pi u/2)}{\Gamma\left(1+\frac{\rho-u}{2}+\delta s\right)}\,ds$$

On applying (1), the result (5) is established.

The integral (6) is established on applying the same procedure and using (3).

FOURIER SERIES

The Fourier series to be obtained are

$$(\sin\theta)^{\rho}\,H_{p,q}^{m,n}\left[z(\sin\theta)^{2\delta}\,\middle|\,\begin{matrix}(a_p,e_p)\\(b_q,f_q)\end{matrix}\right]=\frac{1}{\sqrt{\pi}}\,H_{p+1,q+1}^{m,n+1}\left[z\,\middle|\,\begin{matrix}\left(\frac{1-\rho}{2},\delta\right),(a_p,e_p)\\(b_q,f_q),(-\rho/2,\delta)\end{matrix}\right]$$

$$+\frac{2}{(\pi)^{\frac{1}{2}}}\sum_{r=1}^{\infty}H_{p+2,q+2}^{m,n+2}\left[z\,\middle|\,\begin{matrix}\left(\frac{1-\rho}{2},\delta\right),(-\rho/2,\delta),(a_p,e_p)\\(b_q,f_q),\left(-\frac{\rho+u}{2},\delta\right),\left(-\frac{\rho-u}{2},\delta\right)\end{matrix}\right]\times$$

$$\times\cos\frac{\pi r}{2}\cos r\theta, \tag{7}$$

$$(\sin\ \theta)^{\rho}\, H_{p,\,q}^{m,\,n}\left[\, z\ (\sin\ \theta)^{2\delta} \,\middle|\, \begin{matrix}(a_p,\ e_p)\\ (b_q,\ f_q)\end{matrix}\,\right]$$

$$= \frac{2}{(\pi)^{\frac{1}{2}}} \sum_{r=1}^{\infty} H_{p+2,\,q+2}^{m,\,n+2}\left[\, z \,\middle|\, \begin{matrix}\left(\frac{1-\rho}{2},\ \delta\right),\ (-\rho/2,\ \delta),\ (a_p,\ e_p)\\ (b_q, f_q),\ \left(-\frac{\rho+u}{2},\ \delta\right),\ \left(-\frac{\rho-u}{2},\ \delta\right)\end{matrix}\,\right] \times$$

$$\times\ \sin \frac{\pi\, r}{2}\ \ \sin r\,\theta, \tag{8}$$

where δ is a positive number and

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j < 0, \qquad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv K > 0,$$

$$|\ \arg z\ |\ \ \tfrac{1}{2} K < \pi,\ Re\ 2\ \delta\ b_j/f_j > 1 - \rho\ \ (j = 1, \ldots\ldots, m), \qquad 0 < \theta < \pi.$$

Proof

To establish (7), let

$$f(\theta) = (\sin\ \theta)^{\rho}\, H_{p,\,q}^{m,\,n}\left[\, z\,(\sin\ \theta)^{2\delta} \,\middle|\, \begin{matrix}(a_p,\ e_p)\\ (b_q,\ \ f_q)\end{matrix}\,\right] = \frac{C_0}{2} + \sum_{r=1}^{\infty} C_r \cos r\theta. \tag{9}$$

Equation (9) is valid since $f(\theta)$ is continuous and of bounded variation in the open interval $(0,\ \pi)$, when $\rho > 0$.

Multiplying both sides of (9) by $\cos\ (u\,\theta)$ and integrating with respect to θ from 0 to π, we get

$$\int_0^{\pi} (\sin\ \ \theta)^{\rho} \cos u\,\theta\ H_{p,\,q}^{m,\ n}\left[\, z\ (\sin\ \theta)^{2\delta} \,\middle|\, \begin{matrix}(a_p,\ e_p)\\ (b_q,\ f_q)\end{matrix}\,\right] d\theta$$

$$= \frac{C_0}{2} \int_0^{\pi} \cos\ u\theta\ d\theta\ \ + \sum_{r=1}^{\infty} C_r \int_0^{\pi} \cos r\theta\ \ \cos\ u\theta\ \ d\theta$$

Now using (5) and the orthogonality property of cosine functions, we have

$$C_u = \frac{2}{(\pi)^{\frac{1}{2}}}\ \cos \frac{\pi\ u}{2}\ \ H_{p\,+\,2,\,q\,+\,2}^{m,\ n\ +\ 2}\left[\, z \,\middle|\, \begin{matrix}\left(\frac{1-\rho}{2},\ \delta\right),\ (-\ \rho/2\ ,\ \delta),\ (a_p,\ e_p)\\ (b_q, f_q),\ \left(-\frac{\rho+u}{2},\ \delta\right),\ \left(-\frac{\rho-u}{2},\ \delta\right)\end{matrix}\,\right]$$

From (9) and (10), the result (7) is obtained. (10)

To prove (8), let

$$f(\theta) = (\sin\ \theta)^{\rho}\ H^{m,\ n}_{p,\ q}\left[\ z\ (\sin\ \theta)^{2\delta}\ \middle|\ \begin{matrix}(a_p,\ e_p)\\(b_q,\ f_q)\end{matrix}\right] = \sum_{r=1}^{\infty} C_r \sin r\theta \tag{11}$$

Multiplying both sides of (11) by $\sin (u\ \theta)$ and integrating with respect to θ from 0 to π then using (6) and the orthogonality property of sine functions, we obtain

$$C_u = \frac{2}{(\pi)^{\frac{1}{2}}}\ \sin\frac{\pi u}{2}\ \ H^{m,\ n\ +\ 2}_{p+2,q+2}\left[\ z\ \middle|\ \begin{matrix}\left(\frac{1-\rho}{2}, \delta\right), (-\rho/2, \delta), (a_p, e_p)\\ (b_q, f_q), \left(-\frac{\rho+u}{2}, \delta\right), \left(-\frac{\rho-u}{2}, \delta\right)\end{matrix}\right] \tag{12}$$

From (11) and (12), the formula (8) follows immediately.

PARTICULAR CASES

In (7), assuming δ as a positive integer, putting $e_j = f_i = 1$ ($j = 1, \ldots\ldots,$ p ; $i = 1, \ldots\ldots, q$), using the formula

$$H^{m,\ n}_{p,\ q}\left[\ z\ \middle|\ \begin{matrix}(a_p,\ 1)\\(b_q,\ 1)\end{matrix}\right] = G^{m,\ n}_{p,\ q}\left[\ z\ \middle|\ \begin{matrix}a_p\\b_q\end{matrix}\right],$$

and simplifying with the help[1] of (1), (4) and (9), we get a result recently obtained by Bajpai[9], viz.

$$(\sin\ \theta)^{\rho}\ G^{m,\ n}_{p,\ q}\left[\ z\ (\sin\ \theta)^{2\delta}\ \middle|\ \begin{matrix}a_p\\b_q\end{matrix}\right] = \frac{1}{(\pi\ \delta)^{\frac{1}{2}}}\ G^{m,\ n\ +\ \delta}_{p+\delta,q+\delta}\left[\ z\ \middle|\ \begin{matrix}\Delta\left(\delta, \frac{1-\rho}{2}\right), a_p\\ b_q, \Delta\ (\delta,\ -\ \rho/2)\end{matrix}\right]$$

$$+\ \frac{2}{(\pi\ \delta)^{\frac{1}{2}}}\sum_{r=1}^{8} G^{m,\ n\ +\ 2\ \delta}_{p+2\delta,\ q+2\delta}\left[\ z\ \middle|\ \begin{matrix}\Delta\ (\ 2\ \delta, -\ \rho), a_p\\ b_q, \Delta\left(\delta, -\ \frac{\rho+r}{2}\right), \Delta\left(\delta, -\frac{\rho-r}{2}\right)\end{matrix}\right]$$

$$\times \cos\ \frac{\pi r}{2}\ \cos\ r\theta, \tag{13}$$

where $2\ (m + n) > p + q$, $|\arg z| < (m + n - \frac{1}{2}\ p - \frac{1}{2}\ q)\ \pi$,

$\mathrm{Re}\ (2\ \delta\ b_j) > -\ \rho - 1\ (j = 1, \ldots\ldots, m)$, $0 < \theta < \pi$.

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REFERENCES

1. ERDELYI, A., "Higher Transcedental Functions," Vol. 1, (McGraw-Hill, New York), 1953.
2. CARLSON, B. C. & GREIMAN, W. H., *Duke Math. J.*, **33** (1966), 41.
3. MACROBERT, T. M., *Math. Z.*, **71** (1959), 143.

4. MACROBERT, T. M., *Math. Z.*, 75 (1961), 79.

5. JAIN, R. N., *Math. Japan*, 10 (1965),101.

6. KESARWANI, R. N., *Compositio Math.*, 17 (1966), 149.

7. BAJPAI, S. D., *Gaz. Mat.* (Lisboa), 28 (1967), 40.

8. BAJPAI, S. D., *Proc. Camb., Phil.Soc.*, 65 (1969), 703.

9. BAJPAI, S. D., "Fourier series for G-functions". (Under communication).

10. NIELSEN, N., "Handbuch der Theoric der Gamma-function", (Leipzig), 1906.

11. FOX,C., *Trans. Amer. Math. Soc.*, 98 (1961), 395.

12. BRAAKSMA, B. L.J., *Compositio Math.*, 15 (1963), 239.

13. RAINVILLE, E.D., "Special Functions" (McMillan & Co. Ltd., New York), 1967.