INTRINSIC PROPERTIES OF STEADY DIABATIC COMPLEX-LAMELLAR GAS FLOWS

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Considering the Beltrami surfaces of revolution obtained by revolving a system of confocal hyperbolae in the meridian plane, various kinetic and kinematic properties of steady diabatic complex-lamellar gas flows have been studied.

Several physical and chemical phenomena invalidate the assumption of adiabatic flow in many compressible flow problems. The inviscid non-conducting steady gas flows with energy addition by heat sources are termed diabatic and corresponding to heating processes are thermodynamically reversible. The results from diabatic flow studies provide the basic insight into heat effects which is necessary. Hicks[1] has formulated the fundamental equations governing diabatic steady flows in Crocco's velocity vector field and considered several interesting properties of the flows. The nonlinear character of the equations governing steady diabatic gas flow presents difficulties to obtain exact solutions. Consequently herein adopting the inverse method, viz., assigning the geometric pattern to the Beltrami surface as confocal hyperbolae in the meridian plane, intrinsic properties of complex-lamellar flow have been studied, using the intrinsic equations governing diabatic flows established earlier[2]. From these, possible flows are obtained. Complex-lamellar velocity vector field is characterised and observed that the streamlines form the lines of curvatures. The vortex lines can be geodesics or asymptotic lines on the Beltrami surfaces. The geometry of the Beltrami surfaces, when they form a family of confocal hyperbolae in the meridian plane is completely determined and using these intrinsic decomposition is effected. The necessary compatibility conditions governing the flow are obtained.

FUNDAMENTAL EQUATIONS

The fundamental equations governing steady diabatic gas flow in the absence of extraneous forces, in Crocco's velocity vector field are given below in the usual notation[1]

$$div\left\{ \vec{W} \left(1 - W^2\right)^{\frac{1}{\gamma - 1}} \right\} = q \left(1 + \frac{\gamma + 1}{\gamma - 1} W^2\right) \left(1 - W^2\right)^{\frac{2-\gamma}{\gamma-1}} \tag{1}$$

$$\nabla \operatorname{Log} p_t = \frac{2\gamma}{\gamma - 1} \left(\frac{\vec{W} \wedge \operatorname{Curl} \vec{W} - q \vec{W}}{1 - W^2} \right) \tag{2}$$

$$T_t = T \left(1 - W^2\right)^{-1} \tag{3}$$

$$p_t = p\,(1 - W^2)^{\gamma/1-\gamma} \tag{4}$$

$$q = \frac{\overrightarrow{W}}{2} \cdot \nabla \log T_t = \overrightarrow{W} \cdot \nabla \log V_t \tag{5}$$

In addition to these we add the Crocco's vorticity equation for adiabatic gas flow

$$V^2 \overrightarrow{W} \wedge \text{Curl}\,\overrightarrow{W} + W^2 V_t \nabla V_t = C_p \nabla T_t - T \nabla S \tag{6}$$

where $W, q, \gamma, P_t, V_t, C_pT_t, T$ and S are the reduced velocity vector, the heat content the adiabatic exponent, the total pressure, the limiting velocity, the stagnation enthalpy, the temperature and the specific entropy respectively and $W = \left|\overrightarrow{W}\right|$.

GEOMETRIC RESULTS

The single parameter family of surfaces normals to which determine the direction of flow are defined as the Beltrami surfaces, the velocity vector field are known as complex-lamellar or doubly laminar. These do exist in the case of a complex-lamellar flow.

In such cases the velocity vector can be written as

$$\overrightarrow{W} = \bar{\psi}\,(\overrightarrow{r})\,\nabla\,\bar{\phi}\,(\overrightarrow{r}) \tag{7}$$

where $\bar{\psi}\,(r)$ = Constant and $\bar{\phi}\,(r)$ = Constant are known as the distance function and the Beltrami surface. Operating Curl on (7), we get

$$\text{Curl}\,\overrightarrow{W} = \nabla\,\bar{\psi}\,(\overrightarrow{r}) \wedge \nabla\,\bar{\phi}\,(\overrightarrow{r}) \tag{8}$$

This shows that the vortex line is the curve of intersection of the Beltrami and its distance function surface. Using orthogonal property of streamline and vortexline and denoting $\overrightarrow{S}$ as the unit tangent vector to a streamline, we have

$$\overrightarrow{S} \cdot \text{Curl}\,\overrightarrow{S} = 0 \tag{9}$$

which is the condition that the streamlines to be the lines of curvature[3]. In general the normal plane to a streamline contains the vortexlines, which are on the Beltrami surfaces. But by proper choice of the directions of the streamlines and the vortexlines the vortexlines can be proved as geodesics or asymptotic lines on a Beltrami surfaces.

The geometry of the Beltrami surfaces, when they are family of hyperbolae in the meridian plane is given by

$$x = u \cos\theta\,,\quad y = u \sin\theta\,,\quad \alpha\, u^2 - \beta\, z^2 = \delta \tag{10}$$

where α, β are constants to be determined later and δ is the parameter. The normals to the hyperbolae in the meridian plane $\alpha\, u^2 - \beta\, z^2 = \delta$ are the tangents to the streamlines which determine the direction of flow in the meridian plane. Therefore the principal normal vector to the streamline is along the tangent to the hyperbolae, and the binormal is along the parallels perpendicular to the meridian plane.

Considering $\vec{S}$, $\vec{n}$ and $\vec{b}$ as the unit tangent, principal normal and binormal to the streamline in the present problem these correspond to:

$$\left.\begin{array}{ll} (a) & \vec{S} = i_u \dfrac{\alpha u}{t} - i_z \dfrac{\beta z}{t} \\[2ex] (b) & \vec{n} = i_u \dfrac{\beta z}{t} + i_z \dfrac{\alpha u}{t} \\[2ex] (c) & \vec{b} = i_\theta \end{array}\right\} \tag{11}$$

where $t^2 = \alpha^2 u^2 + \beta^2 z^2$

The curvature K, the mean curvature J and the torsion τ of the streamline respectively are given by

$$\left.\begin{array}{ll} (a) & K = \dfrac{(\alpha + \beta)\, \alpha\, \beta u z}{t^3} \\[2ex] (b) & J = \dfrac{2\alpha - \beta}{t} - \dfrac{1}{t^3} (\alpha^3 u^2 - \beta^3 z^2) \\[2ex] (c) & \tau^2 = \dfrac{\alpha + \beta}{t^3} \alpha \beta u^2 \left\{ \alpha - \dfrac{\alpha\, \beta\, (\alpha + \beta) u z}{t^3} \right\} \end{array}\right\} \tag{12}$$

INTRINSIC DECOMPOSITION

We shall make use of the basic intrinsic equations established in our earlier investigations[2] to study the kinematic and kinetic properties of the flows described above, when the Beltrami surfaces are hyperbolae in the meridian plane.

The fundamental equations (1) and (2) in intrinsic form are given by[2]

$$J = \frac{q\,(1 + \lambda W^2) + (\lambda W^2 - 1) \dfrac{dW}{ds}}{W(1 - W^2)} \tag{13}$$

$$\left.\begin{array}{ll} (a) & \dfrac{d}{ds} \log p_t = \dfrac{2\gamma W q}{(\gamma - 1)(1 - W^2)} \\[2ex] (b) & \dfrac{d}{dn} \log p_t = \dfrac{-2\gamma W}{(\gamma - 1)(1 - W^2)} \left(kW - \dfrac{dW}{dn} \right) \\[2ex] (c) & \dfrac{d}{db} \log p_t = \dfrac{2\gamma W}{(\gamma - 1)(1 - W^2)} \dfrac{dW}{db} \end{array}\right\} \tag{14}$$

where $\frac{d}{ds}$, $\frac{d}{dn}$, $\frac{d}{db}$ are the intrinsic derivatives along the streamlines, their principal normals, binormals and $\lambda = \frac{\gamma+1}{\gamma-1}$.

Making use of (11*a*) and (12*b*) in (13), we obtain the following

$$\frac{2\alpha-\beta}{t} - \frac{\alpha^3u^2-\beta^3z^2}{t^3} = \frac{q\,(1+\lambda W^2)+(\lambda W^2-1)\left(\frac{\alpha u}{t}\frac{\partial W}{\partial u}-\frac{\beta z}{t}\frac{\partial W}{\partial z}\right)}{W(1-W^2)} \tag{15}$$

For Chaplygin's adiabatic gas $\lambda = 0$, (15) simplifies to

$$\left(2\alpha-\beta\right) - \frac{1}{t^2}\left(\alpha^3u^2-\beta^3z^2\right) = \frac{\alpha u\frac{\partial W}{\partial u} - \beta z\frac{\partial W}{\partial z}}{W\,(W^21)} \tag{16}$$

Writing the Lagrange's system of auxiliary equation, we get

$$\frac{du}{\frac{\alpha u}{W\,(W^2-1)}} = \frac{dz}{\frac{-\beta z}{W(W^2-1)}} = \frac{dW}{(2\alpha-\beta)-\left(\frac{\alpha^3u^2-\beta^3z^2}{t^2}\right)} \tag{17}$$

An intermediate integral of (17) is

$$u^{\beta}z^{\alpha} = C \tag{18}$$

where C is constant of integration.

The general solution of (17) can be obtained by using (18) as

$$\log(W^2-1) = \log u^{2(2-\beta/\alpha)} + \chi\,(u^{\beta}z^{\alpha}) + \frac{2}{\alpha}\int\frac{\beta^3c^{2/\alpha}-\alpha^3u^{(2+2\beta/\alpha)}}{u\,(\alpha^2u^2+2\beta/\alpha+\beta^2c^{2/\alpha})}\,du \tag{19}$$

which determines the velocity for adiabatic flow.

Forming the scalar product of (2) by $\vec{S}$, $\vec{n}$ and $\vec{b}$ we obtain the following

$$\frac{1}{t}\left(\alpha u\frac{\partial}{\partial u} - \beta z\frac{\partial}{\partial z}\right)\log p_t = \frac{2\gamma Wq}{(\gamma-1)(1-W^2)} \tag{20}$$

$$\left(\beta z\frac{\partial}{\partial u} + \alpha u\frac{\partial}{\partial z}\right)\log p_t = \frac{-2\gamma W}{(\gamma-1)(1-W^2)}\left\{\frac{\alpha\beta Wuz\,(\alpha+\beta)}{t^2} - \left(\beta z\frac{\partial W}{\partial u} + \alpha u\frac{\partial W}{\partial z}\right)\right\} \tag{21}$$

$$\frac{\partial p_t}{\theta} = \frac{\partial W}{\partial\theta} = 0 \tag{22}$$

These momentum equations can also be written in intrinsic form as

$$\frac{1}{tp}\left(\alpha u \frac{\partial p}{\partial u} - \beta z \frac{\partial p}{\partial z} \right) = \frac{-2\gamma W}{(\gamma-1)(1-W^2)}\left(q + \frac{\alpha u}{t}\frac{W}{\partial u} - \frac{\beta z}{t}\frac{\partial W}{\partial z} \right) \tag{23}$$

$$\beta z \frac{\partial p}{\partial u} + \alpha u \frac{\partial p}{\partial z} = \frac{-2\gamma W^2 p\alpha\beta uz(\alpha+\beta)}{(\gamma-1)(1-W^2)t^2} \tag{24}$$

$$\frac{\partial p}{\partial \theta} = 0 \tag{25}$$

Operating Curl on (2), i.e. eliminating P_t and using (12) we obtain the integrability condition

$$\frac{\alpha\beta uzWq(\alpha+\beta)}{t^2(W^2-1)} - \left(\beta z \frac{\partial}{\partial u} + \frac{\partial}{\partial z} \right)\left(\frac{qW}{W^2-1} \right)$$

$$- \frac{W\alpha\beta\delta}{(W^2-1)t^3}\left\{ \frac{\alpha\beta uz\dot{W}}{t^2}(\alpha+\beta) - \left(\beta z \frac{\partial W}{\partial u} + \alpha u \frac{\partial W}{\partial z} \right) \right\}$$

$$+ \left(\alpha u \frac{\partial}{\partial u} - \beta z \frac{\partial}{\partial z} \right)\left\{ \frac{W^2\alpha\beta uz(\alpha+\beta) - Wt^2\left(\beta z \frac{\partial W}{\partial u} + \alpha u \frac{\partial W}{\partial z} \right)}{t^3(W^2-1)} \right\} = 0 \tag{26}$$

From these adiabatic case can be discussed as a special case.

Writing the Lagrange's system of auxiliary equations for (20) we get

$$\left(du \Big/ \frac{\alpha u}{t} \right) = \left(dz \Big/ -\frac{\beta z}{t} \right) = \left(d(\log p_t) \Big/ \frac{2\gamma qW}{(\gamma-1)(1-W^2)} \right) \tag{27}$$

Since the intermediate integral for (27) is the same as (18) and W is given by a similar relation as that of (19), for the given distribution of heat, the general solution for p_t can be written as

$$\log p_t = \int \frac{2\gamma tqW}{(\gamma-1)(1-W^2)\alpha u} du + \phi(z^\alpha u^\beta) \tag{28}$$

Using this we can obtain an analytic value of the gas dynamic pressure p from (4).

Making use of (11*a*) in (5) we obtain the analytic solutions V_t and T_t as

$$\log V_t = \int \frac{qt}{W\alpha u} du + \varphi_1(u^\beta z^\alpha) \tag{29}$$

$$\log T_t = \int \frac{2tq}{u\alpha W} du + \varphi_2(u^\beta z^\alpha) \tag{30}$$

From (29), (30) and (3) we can evaluate the temperature and the actual velocity $V = WV_t$, which hold along an individual streamline.

In the case of adiabatic steady gas flow, the vorticity and energy equations are given by

$$V^2_t\, \vec{Wb}.\, Curl\, \vec{W} = W^2\, V_t \left(\beta z \frac{\partial V_t}{\partial u} + \alpha u \frac{\partial V_t}{\partial z} \right) + T \left(\beta z \frac{\partial S}{\partial u} + \alpha u \frac{\partial S}{\partial z} \right) -$$

$$- C_p \left(\beta z \frac{\partial T_t}{\partial u} + \alpha u \frac{\partial T_t}{\partial z} \right) \tag{31}$$

$$\alpha u \frac{\partial S}{\partial u} - \beta z \frac{\partial S}{\partial z} = 0 \tag{32}$$

Therefore the specific entropy is given by

$$S = f\,(u^{\beta} z^{\alpha}) = S \left\{ (x^2 + y^2)^{\beta/2} z^{\alpha} \right\} \tag{33}$$

The compatibility condition for adiabatic case can be deduced from (26). Hence the adiabatic phenomenon can be discussed as special cases of this investigation.

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