SHOCKS IN RADIATION GAS DYNAMICS

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The jump conditions across a three dimensional curved shock in radiation gas dynamics have been derived. The successive approximations have been used for different cases depending upon the value of radiation pressure number. The results are obtained in a form very convenient to treat the further problems in the subject.

Gross[1], Koch[2], Marshak[3], Pai[4] and Sen[5] have considered the effects of radiations on shock-wave structure and Rankine-Hugoniot conditions. Recently Koch and Gross[6] have obtained the jump conditions for hydrogen for plasmas which are optically thin. Our aim, in this paper, is to develop the results for a three dimensional curved-shock in an optically thick medium. We have derived relations analogous to Rankine-Hugoniot relations in ordinary (i.e. without radiation effects) gas dynamics. To achieve the above, we have used as parameters the density shock strength and either the ordinary normal Mach number or the effective normal Mach number in front of the shock. The choice in the above two depends upon the different cases considered.

To solve the equation determining the density shock strength which is of higher degree than can be handled easily, we have used the method of successive approximations as suggested by Pai[4]. We have obtained the values of the flow variables behind the shock completely in terms of the known quantities in front of the shock-surface for different cases depending upon the values of the radiation pressure number in front of the shock. The graph showing the variations of the density shock strength with the effective normal Mach number in front has also been plotted and the help of computer has also been taken to obtain the value of density shock strength upto third approximations, given in a tabular form.

Let the shock configuration be given by

$$x_i = x_i \, (y^{\alpha}, t) \tag{1}$$

where x_i ($i=1, 2, 3$) are the regular cartesian coordinates and y^{α} (α = I, II) are the Guassian coordinates of any point 'S' of the shock surface. We shall denote by $x_{i,\alpha}$ and n_i the tangent and the unit normal vectors to the shock surface[7]. Let u_i, p, p_R, ρ, T and E_R denote the velocity components, pressure, radiation pressure, density, temperature and the radiation energy density of the gas respectively. The components v_i of the relative velocity of the gas with respect to the shock is given by $v_i = u_i - V_i$, where V_i is the velocity of the shock.

Proceeding in a manner similar to Bleakney & Taub[8], Pant & Misra[9] and considering the effects of radiations in an optically thick medium, the conservation of mass, momentum and energy give the following relations, which connect the flow variables on the two sides of the shock surface.

$$\rho_2 v_{2n} = \rho_1 v_{1n} = m , \tag{2}$$

$$m v_{2n} + p_{2t} = m v_{1n} + p_{1t} , \tag{3}$$

$$v_{2\alpha} = v_{1\alpha} , \tag{4}$$

$$\frac{1}{2} v^2_{2n} + C_v T_2 + \frac{E_{2R}}{\rho_2} + \frac{p_{2t}}{\rho_2} = \frac{1}{2} v^2_{1n} + C_v T_1 + \frac{E_{1R}}{\rho_1} + \frac{p_{1t}}{\rho_1} , \tag{5}$$

where

$$\left. \begin{array}{l} v_n = v_i n_i , \; v_\alpha = v_i x_{i,\alpha}, \; p_t = p + p_R , \quad p = \rho R T, \\ p_R = \frac{1}{3} a_R T^4, \; E_R = a_R T^4 , \quad C_v T = \frac{1}{r-1} \frac{p}{\rho} \end{array} \right\} \tag{6}$$

and r, Cv, R, a_R, are respectively the ratio of the specific heats, specific heat at constant volume, gas constant and the Stefan-Boltzmann constant. In the above relations subscript 1(2) denotes quantities in front of (behind) the shock and without any subscript refer to either side of the shock. The effective normal Mach number in radiation gas dynamics is given[8] by

$$M^2_{en} = \frac{r \{ 1 + 12 (r-1) R_p \} M^2_n}{r + 20 (r-1) R_p + 16 (r-1) R^2_p} , \tag{7}$$

where M_n is the normal shock Mach number and is defined as

$$M^2_n = \frac{v^2_n}{rRT} .$$

DERIVATION OF THE JUMP CONDITIONS

An attempt to derive relations analogous to Rankine-Hugoniot relations has been made. The density shock strength δ is defined as

$$\delta = \frac{\rho_2 - \rho_1}{\rho_1} , \tag{8}$$

which combining with (2) gives

$$v_{2n} = \frac{1}{1+\delta} v_{1n} . \tag{9}$$

The relative velocity vector of the gas can be expressed as

$$v_i = v_n n_i + v^\alpha x_{i,\alpha} , \tag{10}$$

where $v^{\alpha\beta} = a^{\alpha\beta} v_\beta$, and $a_{\alpha\beta} = x_{i,\alpha} x_{i,\beta}$ is the metric tensor of the shock-surface[5]. Using (4) and (9), (10) transforms to

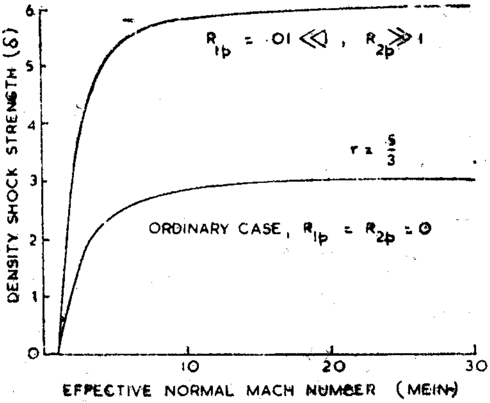

Fig. 1—Variation of density shock strength with effective normal Mach number.

$$v_{2i} - v_{1i} = -\frac{\delta}{1+\delta} v_{1n} n_i . \tag{11}$$

We take the following non-dimensional parameters

$$T^* = \frac{RT}{v^2_{in}}, \quad P = T^*_1 = \frac{1}{\gamma M^2_{in}}, \quad Q = \frac{a_R v^6{}_{in}}{R^4 \rho_1} \tag{12}$$

and

$$R_p = \frac{p_R}{p} = \frac{1}{3} \frac{Q}{1+\delta} (T^*)^3 , \tag{13}$$

where R_p is called the radiation pressure number. From (2), (6), (9) and (12), the momentum and the energy laws [i.e., eqns. (3) and (5)] transform respectively to

$$\frac{1}{1+\delta} + (1+\delta) T^*_2 + \frac{1}{3} Q (T^*_2)^4 = 1 + P + \frac{1}{3} Q P^4 \tag{14}$$

and

$$\frac{1}{2(1+\delta)} + \frac{1+\delta}{\gamma - 1} T^*_2 + Q (T^*_2)^4 + (1+\delta) T^*_2 + \frac{1}{3} Q (T^*_2)^4$$
$$= (1+\delta) \left(\frac{1}{2} + \frac{\gamma}{\gamma - 1} P + \frac{4}{3} Q P^4 \right) \tag{15}$$

From the above relations, using (13) we get the following relations

$$\frac{1}{1+\delta} + T^*_2 (1+\delta)(1+R_{2p}) = 1 + P + PR_{1p} \tag{16}$$

and

$$-\frac{1}{2(1+\delta)} + T^*_2 (1+\delta) \left(\frac{1}{\gamma - 1} + 3R_{2p} \right) = (1+\delta) \left(\frac{1}{2} + \frac{\gamma}{\gamma - 1} P + 4PR_{1p} \right) -$$
$$- (1 + P + PR_{1p}) \tag{17}$$

Eliminating T^*_2 from the above two relations we obtain

$$\left(\frac{1}{1+\delta} - 1 \right) \left[\frac{1}{1+\delta} (q^2 + 7R_{2p}) - \{ 1 + R_{2p} + P (1 + q^2 + 8R_{2p})(1 + R_{1p}) \} \right] +$$
$$+ P (7 - q^2)(R_{1p} - R_{2p}) = 0 , \tag{18}$$

where
$$q^2 = \frac{\gamma + 1}{\gamma - 1} .$$

Equation (18) can be put in the following convenient form

$$\delta [1 - (1+\delta) (q^2 + 7R_{2p})^{-1} \{ 1 + R_{2p} + (1 + q^2 + 8R_{2p}) P_{2e} \}] = 0 \tag{19}$$

where

$$P_e = (1 + R_{1p}) \; f(R_p) \, P , \tag{20}$$

$$f(R_p) = \frac{g(R_p)(1+\delta)-1}{\delta}, \tag{21}$$

and

$$g(R_p) = \frac{(1+R_p)(1+q^2+8R_{1p})}{(1+R_{1p})(1+q^2+8R_p)}, \tag{22}$$

P_e is called the effective value of P in radiation gas dynamics. Equation (19) gives two roots of δ, one is $\delta = 0$, which refers to no shock conditions. The physically important solution is the other root which gives the density of the gas behind the shock-wave.

This value is given by

$$\delta = \frac{2(1-P_{2e}r_{2e})}{2P_{2e}r_{2e}+r_{2e}-1}, \tag{23}$$

where

$$r_e = \frac{4(r-1)R_p+r}{3(r-1)R_p+1} \tag{24}$$

and is called the effective ratio of the specific heats. Equation (18) also gives the following relations which we shall use later on.

$$R_{2p} - R_{1p} = \frac{\delta(1+\delta)(1+R_{1p})K - \delta(q^2+7R_{1p})}{(1+\delta)(7P-q^2P-K\delta)+7\delta} \tag{25}$$

and

$$K(1+R_{2p})\delta^2 - [(1+R_{2p})(q^2-K)+(7-q^2)\{R_{2p}+PR_{2p}-PR_{1p}\}]\delta - $$
$$- P(7-q^2)(R_{2p}-R_{1p}) = 0, \tag{26}$$

where $\qquad K = 1 + P(1+q^2+8R_{1p}).$

The equation corresponding to (19) in ordinary shocks is simple. Due to the inclusion of radiation effects this equation becomes very complicated. In fact, to determine the values of δ explicitly in terms of known quantities in front of the shock, we have to solve a higher degree equation in δ, as equations (23) and (26) contain R_{2p} and f which themselves are function of δ. To get the solution for different cases offered by different values of R_{1p}, we take the approximations and for this purpose (23) is the most convenient form.

THE CASE WHEN $R_{1p} << 1$

In this case the temperature in front of the shock is not very high and therefore the radiation effects may be neglected in front of the shock. From relations (12) and (16), we obtain

$$T^*_2 = \frac{1+\delta(1+rM^2_{1n})}{rM^2_{1n}(1+\delta)^2(1+R_{2p})} \tag{27}$$

Using the above relation along with (6), (8), (12), (13) and (26) we obtain

$$\frac{T_2}{T_1} = \frac{T^*_2}{P} \doteq \frac{1 + \delta\,(1 + rM^2_{1n})}{(1+\delta)^2\,(1 + R_{2p})}\,,$$

$$\frac{p_2}{p_1} = \frac{\rho_2 T_2}{\rho_1 T_1} \doteq \frac{1 + \delta\,(1 + rM^2_{1n})}{(1+\delta)\;(1 + R_{2p})}\,, \qquad (28)$$

$$\frac{p_{2R}}{p_{1R}} = \left(\frac{T_2}{T_1}\right)^4 \doteq \left\{\frac{1 + \delta\,(1 + rM^2_{1n})}{(1+\delta)^2\,(1 + R_{2p})}\right\}^4,$$

$$\frac{R_{2p}}{R_{1p}} = \frac{p_{2R}}{p_{1R}} \times \frac{p_1}{p_2} \doteq \frac{1}{(1+\delta)} \left\{\frac{1 + \delta\,(1 + rM^2_{1n})}{(1+\delta)^2\,(1 + R_{2p})}\right\}^3, \qquad (29)$$

and

$$K_1\,(1 + R_{2p})\,\delta^2 - \{(1 + R_{2p})\,(q^2 - K_1) + (7 - q^2)\,(1 + P)\,R_{2p}\}\,\delta -$$
$$- P\,(7 - q^2)\,R_{2p} = 0\,, \qquad (30)$$

where
$$K_1 = 1 + P\,(1 + q^2).$$

For this case we have two possibilities. The first one is of weak shock in which $R_{2p} << 1$ and the other one that of a strong shock in which $R_{2p} >> 1$. When $R_{2p} << 1$, relations (28) to (30) reduce to the Rankine-Hugoniot relations in ordinary gas dynamics.

When $R_{2p} >> 1$, then from (24) and (20) we have

$$r_{2e} \cong 4/3 \text{ and } P_{2e} \cong f\,(R_{2p})\,P\,. \qquad (31)$$

As a first approximation we take $R_{1p} \doteq R_{2p}$, then from (21) and (22) we find that functions f and g are nearly equal to unity. Using this fact and (31) in (23) we get $(\delta)_1 = 6$, Pai[10]. For the second approximation, from (20) to (22) we obtain

$$g\;(R_{2p}) \;\cong\; \frac{r}{4\;(r-1)}\,,\; f\,(R_{2p}) \quad \frac{3r + 4}{24\;(r-1)} \text{ and } P_{2e} = \frac{3r+4}{24\,(r-1)}\,P\,. \qquad (32)$$

Using (31) and (32), we get from (23)

$$\delta \cong \frac{18r\,(r-1)\;M^2_{1n} - (3r + 4)}{3r\;(r-1)\;M^2_{1n} + (3r+4)}\,, \qquad (33)$$

which determines the value of δ completely in terms of known quantities. For a very strong shock (i.e., $M^2_{1n} >> 1$), $\delta = 6$ irrespective of the value of r. This is the maximum attainable value of δ, while in ordinary gas dynamics it is 5 for $r = 1{\cdot}4$.

From (29) we have

$$R_{2p} \doteq \left\{\frac{1 + \delta\;(1 + rM^2_{1n})}{(1+\delta)^2}\right\}^{3/4} \times \left\{\frac{R_{1p}}{1+\delta}\right\}^{1/4}. \qquad (34)$$

From (8), (11), (28), (29) and (34), we have for this case the following relations

$$\left.\begin{aligned}
\frac{T_2}{T_1} &\simeq \left\{ \frac{1+\delta\,(1+rM^2{}_{1n})}{(1+\delta)\,R_{1p}} \right\}^{1/4} , \\
\frac{p_2}{p_1} &\simeq (1+\delta) \left\{ \frac{1+\delta\,(1+rM^2{}_{1n})}{(1+\delta)\,R_{1p}} \right\}^{1/4} , \\
\frac{p_{2R}}{p_{1R}} &\cong \frac{1+\delta\,(1+rM^2{}_{1n})}{(1+\delta)\,R_{1p}} , \\
\frac{R_{2p}}{R_{1p}} &\cong \frac{1}{(1+\delta)} \left\{ \frac{1+\delta\,(1+r\,M^2{}_{1n})}{(1+\delta)\,R_{1p}} \right\}^{3/4} , \\
\frac{\rho_2}{\rho_1} &= 1+\delta \text{ and } v_{2i} = v_{1i} - \frac{\delta}{1+\delta}\, v_{1n}\, n_i ,
\end{aligned}\right\} \tag{35}$$

where δ is given by (33).

The above relations determine the flow variables behind the shock in terms of the known flow quantities in front of the shock.

THE CASE WHEN $R_{1p} >> 1$

In this case the temperature of the gas in front of the shock is high and therefore, the radiation effects in front of the shock cannot be neglected and consequently, they cannot be neglected behind as well, i.e., $R_{2p} >> 1$, irrespective of the shock strength.

From (20) to (22) and (24) we have

$$\left.\begin{aligned}
& g\,(R_{2p}) \cong 1 ,\ f\,(R_{2p}) \cong 1 , \\
& r_{2e} \quad 4/3 \text{ and } P_{2e} \simeq R_{1p}/(rM^2{}_{1n})
\end{aligned}\right\} \tag{36}$$

Using the above relations in (23) we obtain

$$\delta = \frac{6\,(M^2{}_{e1n} - 1)}{M^2{}_{e1n} + 6} , \tag{37}$$

where the effective normal Mach number in front of the shock for this case is $M^2{}_{e_1n} = \dfrac{3r\,M^2{}_{1n}}{4\,R_{1p}}$. Using (13), eqn. (16) transforms to

$$(T^*{}_2)^4 + A^{-1}\,T^*{}_2 - A^{-1}\,B = 0 , \tag{38}$$

where $$A^{-1} = \frac{P^3\,(1+\delta)}{R_{1p}} \text{ and } B = \frac{(1+P+PR_{1p})\,(1+\delta) - 1}{(1+\delta)^2} . \tag{39}$$

For $R_{1p} >> 1$, $(T^*{}_2)^4 >> A^{-1}\ T^*{}_2$, Pai[10]. Therefore, from (8), (11) and (38) using (8), (12), (13), (6) and (39) we obtain the following relations.

$$\left.\begin{aligned}
\frac{T_2}{T_1} &\simeq \left\{ 1 + \frac{4}{3}\,\frac{\delta}{1+\delta}\, M^2_{el n} \right\}^{1/4}, \\
\frac{p_2}{p_1} &\cong (1+\delta) \left\{ 1 + \frac{4}{3}\,\frac{\delta}{1+\delta}\, M^2_{el n} \right\}^{1/4}, \\
\frac{p_{2R}}{p_{1R}} &\cong \left\{ 1 + \frac{4}{3}\,\frac{\delta}{1+\delta}\, M^2_{el n} \right\}, \\
\frac{R_{2p}}{R_{1p}} &\cong \frac{1}{(1+\delta)} \left\{ 1 + \frac{4}{3}\,\frac{\delta}{1+\delta}\, M^2_{el n} \right\}^{3/4}, \\
\frac{\rho_2}{\rho_1} &\cong 1+\delta \text{ and } v_{2i} = v_{1i} = v_{1i} - \frac{\delta}{1+\delta}\, v_{1n}\, n_i ,
\end{aligned}\right\} \quad (40)$$

where δ is given by (37).

CASE WHEN R_{1p} IS NOT TOO LARGE COMPARED TO UNITY BUT STILL $(T^*_2)^4 >> A^{-1}\, T^*_2$ HOLDS

Using $R_{1p} \simeq R_{2p}$ as the first approximation, we obtain from (20) to (22) and (24)

$$P_{2e} \cong (1 + R_{1p})\, P \text{ and } r_{2e} \quad \frac{4\,(r-1)\, R_{1p} + r}{3\,(r-1)\, R_{1p} + 1}. \quad (41)$$

Using above relation in (23), we get

$$\delta \quad \frac{2\, r\, M^2_{1n} \{ 3\,(r-1)\, R_{1p} + 1 \} - (1 + R_{1p}) \{ 8\,(r-1)\, R_{1p} + 2r \}}{r\, M^2_{1n} \{ (r-1)\,(1 + R_{1p}) \} + (1 + R_{1p}) \{ 8\,(r-1)\, R_{1p} + 2\, r \}} \quad (42)$$

and from (6), (8), (11) to (13) and (39) we get the following relations

$$\left.\begin{aligned}
\frac{T_2}{T_1} &\cong \left\{ \frac{1 + R_{1p}}{R_{1p}} + \frac{\delta\, r\, M^2_{1n}}{(1+\delta)\, R_{1p}} \right\}^{1/4}, \\
\frac{p_2}{p_1} &\simeq (1+\delta) \left\{ \frac{1 + R_{1p}}{R_{1p}} + \frac{\delta\, r\, M^2_{1n}}{(1+\delta)\, R_{1p}} \right\}^{1/4}, \\
\frac{p_{2R}}{p_{1R}} &\cong \left\{ \frac{1 + R_{1p}}{R_{1p}} + \frac{\delta\, r\, M^2_{1n}}{(1+\delta)\, R_{1p}} \right\}, \\
\frac{R_{2p}}{R_{1p}} &\cong \frac{1}{1+\delta} \left\{ \frac{1 + R_{1p}}{R_{1p}} + \frac{\delta\, r\, M^2_{1n}}{(1+\delta)\, R_{1p}} \right\}^{3/4}, \\
\frac{\rho_2}{\rho_1} &= (1+\delta) \text{ and } v_{2i} = v_{1i} - \frac{\delta}{1+\delta}\, v_{1n}\, n_i ,
\end{aligned}\right\} \quad (43)$$

which give the variables behind the shock in terms of known variables in front of the shock. In the above two cases further approximations are obtained by numerical methods, while in the first case the second approximation obtained gives quite accurate results.

Computed results of variation of shock strength with effective normal Mach number for different radiation pressure numbers in front of the shock upto third approximations are given in Table 1.

TABLE 1

VARIATION OF SHOCK STRENGTH WITH EFFECTIVE NORMAL MACH NUMBER FOR DIFFERENT RADIATION PRESSURE NUMBERS IN FRONT OF THE SHOCK

M_{eln}	δ				
	$R_{1p} = \cdot 01$	$R_{1p} = 0\cdot 1$	$R_{1p} = 1\cdot 0$	$R_{1p} = 1\cdot 5$	$R_{1p} = 2\cdot 0$
5	3·72404528	4·07129861	4·43235970	4·47216702	5·36881924
10	4·80797482	5·14123631	5·43137551	5·46902848	5·71651364
15	5·25150395	5·50128079	5·70091439	5·72735787	5·82751561
20	5·48010731	5·66811848	5·81223107	5·83146573	5·87796212
25	5·61411763	5·76009751	5·86946488	5·88414479	5·90560151
30	5·69989873	5·81672002	5·90307523	5·91470910	5·92260743
35	5·75844193	5·85434247	5·92464829	5·93414212	5·93572903
40	5·80036832	5·88077450	5·93939591	5·94033907	5·94188691

Table 1 shows that the shock strength increases with effective normal Mach number as the radiation pressure number in front of the shock increases.

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