SLOW UNSTEADY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID BETWEEN TWO WAVY WALLS

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(*Received* 10 *April* 1970)

Slow unsteady flow of a viscous, incompressible fluid between two plates with roughness along their length, under the influence of periodic pressure gradient, has been discussed. Integral transform technique has been used to determine pressure and velocity components along and perpendicular to the length of the plates and the pressure. Particular cases of sinusoidal roughness when the phase differences are zero and π have been solved numerically.

The exact solutions of Navier-Stokes' equation for a viscous, incompressible fluid with axially parallel flow through a tube under the influence of periodic pressure gradient have been discussed by Sexl[1] and Uchida[2]; while those of co-axial circular cylinders by Verma[3]. The slow viscous flow between rotating concentric infinite cylinders with axial roughness was discussed by Citron[4] and the problems of flow of non-Newtonian fluids and heat transfer between wavy walls and wavy cylinders have been extensively studied by Bhatnagar & Mohan Rao[5], Bhatnagar & Mathur[6] and Mathur[7]. Recently Verma & Gaur[8] have studied the slow motion of a viscous incompressible fluid in a circular tube with axial roughness under the influence of periodic pressure gradient.

In this paper, the slow unsteady flow of a viscous incompressible fluid between two plates with roughness along their length has been discussed. The roughness has been taken to be small in comparison with the distance between the plates at the mouth of the channel. Integral transform technique has been used to determine the longitudinal and transverse velocity components. An expression for the pressure distribution throughout the length of the channel has also been obtained. Two particular cases of sinusoidal roughness have been discussed numerically.

FORMULATION OF THE PROBLEM

The Navier-Stokes' equations in rectangular coordinate system for a viscous incompressible fluid neglecting the external forces are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w, \tag{3}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

u, v, w are the velocity components along x, y, z directions, ν is the kinematic viscosity, and ρ is the density of the fluid.

Under the assumption of slow motion, from (1) to (4) we have

$$\nabla^2 p = 0 \tag{5}$$

If the axis of x is chosen along the length of the plates and z is measured at right angles to it, we have

$$v = 0, \quad \frac{\partial}{\partial y}\left(\quad\right) = 0 \tag{6}$$

Therefore, (1) to (4) are reduced to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{7}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{8}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{9}$$

Using (5) in (7) we get

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) = \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)^2 u \tag{10}$$

Let

$$U = \frac{au}{\nu}, \quad W = \frac{aw}{\nu}, \quad X = \frac{x}{a},$$

$$Z = \frac{z}{a}, \quad P = \frac{pa^2}{\nu^2\rho} \quad \text{and} \quad T = \frac{t\nu}{a^2}, \tag{11}$$

where a is half the distance between the plates at the mouth of the channel.

Equations (7) to (10) are transformed to

$$\frac{\partial U}{\partial T} = -\frac{\partial P}{\partial X} + \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2}\right), \tag{12}$$

$$\frac{\partial W}{\partial T} = -\frac{\partial P}{\partial Z} + \left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Z^2}\right), \tag{13}$$

$$\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0, \tag{14}$$

and

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right)^2 U = \frac{\partial}{\partial T}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right) U \tag{15}$$

The boundary conditions are

$$U = W = 0 \quad \text{at} \quad Z = 1 + \epsilon N_1(X) \quad \text{and} \quad Z = -1 + \epsilon N_2(X),$$
$$T > 0, \;\; X > 0$$

where $\epsilon \ll 1$ is the roughness parameter.

Let

$$\left.\begin{aligned} P(X, Z, T) &= P_0(X, T) + P'(X, Z, T), \\ W(X, Z, T) &= W'(X, Z, T), \\ U(X, Z, T) &= U_0(Z, T) + U'(X, Z, T) \end{aligned}\right\} \tag{16}$$

where the primed quantities are the variations caused by the roughness and P_0 and U_0 are the quantities for the case of smooth plates, given by

$$\frac{\partial P_0}{\partial Z} = 0, \tag{17}$$

and
$$\frac{\partial U_0}{\partial T} = -\frac{\partial P_0}{\partial X} + \frac{\partial^2 U_0}{\partial Z^2} \tag{18}$$

Let
$$-\frac{\partial P_0}{\partial X} = K \cos nT = Re\left[K e^{inT}\right] \tag{19}$$

and
$$U_0 = f(Z) \cos nT = Re\left[f(Z)\, e^{inT}\right] \tag{20}$$

where Re means the real part.

From (18) to (20), we have

$$f''(Z) - in f(Z) = -K, \tag{21}$$

the solution of which gives

$$U_0(Z, T) = Re\left[\frac{K}{m^2}\left(1 - \frac{\cosh mZ}{\cosh m}\right) e^{inT}\right] \tag{22}$$

where
$$m = \sqrt{(in)},$$

which for very slow oscillations, reduces to

$$U_0(Z, T) = \frac{K}{2}\left(1 - Z^2\right)\cos nT = Re\left[\frac{K}{2}\left(1 - Z^2\right) e^{inT}\right] \tag{23}$$

From (12) to (16) we have

$$\frac{\partial U'}{\partial T} = -\frac{\partial P'}{\partial X} + \left(\frac{\partial^2 U'}{\partial X^2} + \frac{\partial^2 U'}{\partial Z^2}\right), \tag{24}$$

$$\frac{\partial W'}{\partial T} = -\frac{\partial P'}{\partial Z} + \left(\frac{\partial^2 W'}{\partial X^2} + \frac{\partial^2 W'}{\partial Z^2}\right), \tag{25}$$

$$\frac{\partial U'}{\partial X} + \frac{\partial W'}{\partial Z} = 0, \tag{26}$$

and $$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Z^2}\right)^2 U' = \frac{\partial}{\partial T}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Z^2}\right) U', \tag{27}$$

under the boundary conditions

$$\left.\begin{array}{ll} U' = -U_0,\ W' = 0 \ \text{at}\ Z = 1 + \epsilon N_1(X) \ \text{and}\ Z = -1 + \epsilon N_2(X), & T > 0,\ X > 0 \\ U' = 0 \ \text{at}\ -1 + \epsilon N_2(X) < Z < 1 + \epsilon N_1(X), & T > 0,\ X \doteq 0 \end{array}\right\} \tag{28}$$

METHOD OF SOLUTION

Following (20), we assume

$$\begin{aligned} U'(X, Z, T) &= \overline{U}(X, Z)\cos nT \\ &= Re\left[\overline{U}(X,Z)e^{inT}\right] \end{aligned} \tag{29}$$

Equation (27) is therefore reduced to

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Z^2}\right)^2 \overline{U} = in\left(\frac{\partial^2\overline{U}}{\partial X^2}+\frac{\partial^2\overline{U}}{\partial Z^2}\right) \tag{30}$$

Let $$\frac{\partial^2\overline{U}}{\partial X^2}+\frac{\partial^2\overline{U}}{\partial Z^2} = f(X, Z) \tag{31}$$

From (30) and (31), we have

$$\frac{\partial^2 f}{\partial X^2}+\frac{\partial^2 f}{\partial Z^2} = inf. \tag{32}$$

Let $$F(Z, \xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(Z, X)\sin(\xi X)\, dX, \tag{33}$$

where $F(Z, \xi)$ is the Fourier sine transform of $f(Z, X)$.

Taking Fourier sine transform of (32) we have

$$\frac{d^2F}{dZ^2} - (\xi^2 + in) F = 0, \tag{34}$$

the solution of which is

$$F(Z, \xi) = A'(\xi)\, e^{-bZ} + B'(\xi)\, e^{bZ}, \tag{35}$$

where $b = (\sqrt{\xi^2 + in})$ and $A'(\xi)$ and $B'(\xi)$ are the constants of integration.

Taking the Fourier sine transform of (31), we have

$$\frac{d^2U}{dZ^2} - \xi^2 U = F(Z, \xi) \tag{36}$$

where U is the Fourier sine transform of $\overline{U}$.

From (35) and (36), we write

$$\frac{d^2U}{dZ^2} - \xi^2 U = A'(\xi)\, e^{-bZ} + B'(\xi)\, e^{bZ}, \tag{37}$$

the complete solution of which is

$$U(Z,\xi) = A(\xi)\,e^{-bZ} + B(\xi)\,e^{bZ} + C(\xi)\,e^{-\xi Z} + D(\xi)\,e^{\xi Z}, \tag{38}$$

where $A(\xi)$, $B(\xi)$, $C(\xi)$ and $D(\xi)$ are the constants of integration.

Taking inverse Fourier sine transform of U, we have

$U'(X, Z, T) =$

$$Re\,\sqrt{\frac{2}{\pi}}\int_0^\infty \left[A(\xi)\,e^{-bZ} + B(\xi)\,e^{bZ} + C(\xi)\,e^{-\xi Z} + D(\xi)\,e^{\xi Z}\right]\sin(\xi X)\,e^{inT}\,d\xi. \tag{39}$$

From (26) we get

$$\frac{\partial W'}{\partial Z} = -\frac{\partial V'}{\partial X}$$

$$= Re\left(-\sqrt{\frac{2}{\pi}}\right)\int_0^\infty \xi\left[A(\xi)\,e^{-bZ} + B(\xi)\,e^{bZ} + C(\xi)\,e^{-\xi Z} + D(\xi)\,e^{\xi Z}\right]\cos(\xi X)\,e^{inT}\,d\xi,$$

which gives

$$W'(X, Z, T) = Re\left(-\sqrt{\frac{2}{\pi}}\right)$$

$$\int_0^\infty\left[-\frac{\xi}{b}A(\xi)\,e^{-bZ} + \frac{\xi}{b}B(\xi)\,e^{bZ} - C(\xi)\,e^{-\xi Z} + D(\xi)\,e^{\xi Z}\right]\cos(\xi X)\,e^{inT}\,d\xi \tag{40}$$

and from (24) and (25) using (39) and (40), we get

$$P'(X, Z, T) = Re\left(\sqrt{\frac{2}{\pi}}\,in\right).$$

$$\int_0^\infty \frac{1}{\xi}\left[C(\xi)\,e^{-\xi Z} + D(\xi)\,e^{\xi Z}\right]\cos(\xi X)\,e^{inT}\,d\xi + C \tag{41}$$

where C is the constant of integration.

Using the boundary conditions (28) in (39) and (40), we get

$$-\frac{K}{2}\left[1-\left\{1+\epsilon N_1(X)\right\}^2\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty\left[A(\xi)\exp\left\{-b\left(1+\epsilon N_1(X)\right)\right\}\right.$$

$$+ B(\xi)\exp\left\{b\left(1+\epsilon N_1(X)\right)\right\} + C(\xi)\exp\left\{-\xi\left(1+\epsilon N_1(X)\right)\right\}$$

$$\left.+ D(\xi)\exp\left\{\xi\left(1+\epsilon N_1(X)\right)\right\}\right]\sin(\xi X)\,d\xi \tag{42}$$

$$0 = -\sqrt{\frac{2}{\pi}} \int_0^\infty \Big[-\frac{\xi}{b} A(\xi) \exp\Big\{ -b\Big(1 + \epsilon N_1(X)\Big)\Big\} +$$

$$+ \frac{\xi}{b} B(\xi) \exp\Big\{ b\Big(1 + \epsilon N_1(X)\Big)\Big\} - C(\xi) \exp\Big\{ -\xi\Big(1 + \epsilon N_1(X)\Big)\Big\} +$$

$$+ D(\xi) \exp\Big\{ \xi \Big(1 + \epsilon N_1(X)\Big)\Big\}\Big] \cos(\xi X)\, d\xi, \tag{43}$$

$$-\frac{K}{2}\Big[1 - \Big\{-1 + \epsilon N_2(X)\Big\}^2\Big] = \sqrt{\frac{2}{\pi}} \int_0^\infty \Big[A(\xi) \exp\Big\{-b\Big(-1 + \epsilon N_2(X)\Big)\Big\}$$

$$+ B(\xi) \exp\Big\{b\Big(-1 + \epsilon N_2(X)\Big)\Big\} + C(\xi) \exp\Big\{-\xi\Big(-1 + \epsilon N_2(X)\Big)\Big\} +$$

$$+ D(\xi) \exp\Big\{\xi\Big(-1 + \epsilon N_2(X)\Big)\Big\}\Big] \sin(\xi X)\, d\xi \tag{44}$$

$$0 = -\sqrt{\frac{2}{\pi}} \int_0^\infty \Big[-\frac{\xi}{b} A(\xi) \exp\Big\{-b\Big(-1 + \epsilon N_2(X)\Big)\Big\} +$$

$$+ \frac{\xi}{b} B(\xi) \exp\Big\{b\Big(-1 + \epsilon N_2(X)\Big)\Big\} - C(\xi) \exp\Big\{-\xi\Big(-1 + \epsilon N_2(X)\Big)\Big\} +$$

$$+ D(\xi) \exp\Big\{\xi\Big(-1 + \epsilon N_2(X)\Big)\Big\}\Big] \cos(\xi X)\, d\xi. \tag{45}$$

Let

$$A(\xi) = A_0(\xi) + \epsilon A_1(\xi) + \ldots\ldots$$

and similar expansions for $B(\xi)$, $C(\xi)$ and $D(\xi)$, (46)

Substituting (46) in (42) to (45) and equating the coefficients of like powers of ϵ, we have From coefficients of ϵ^0

$$0 = \sqrt{\frac{2}{\pi}} \int_0^\infty \Big[A_0(\xi)\, e^{-b} + B_0(\xi)\, e^{b} + C_0(\xi)\, e^{-\xi} + D_0(\xi)\, e^{\xi}\Big] \sin(\xi X)\, d\xi,$$

$$0 = -\sqrt{\frac{2}{\pi}} \int_0^\infty \Big[-\frac{\xi}{b} A_0(\xi)\, e^{-b} + \frac{\xi}{b} B_0(\xi)\, e^{b} - C_0(\xi)\, e^{-\xi} + D_0(\xi)\, e^{\xi}\Big] \cos(\xi X)\, d\xi,$$

$$0 = \sqrt{\frac{2}{\pi}} \int_0^\infty \Big[A_0(\xi)\, e^{b} + B_0(\xi)\, e^{-b} + C_0(\xi)\, e^{\xi} + D_0(\xi)\, e^{-\xi}\Big] \sin(\xi X)\, d\xi,$$

$$0 = -\sqrt{\frac{2}{\pi}} \int_0^\infty \left[-\frac{\xi}{b} A_0(\xi) e^{b} + \frac{\xi}{b} B_0(\xi) e^{-b} - C_0(\xi) e^{\xi} + D_0(\xi) e^{-\xi} \right] \cos(\xi X)\, d\xi .$$

Inverting the above four equations by Fourier sine and cosine integral theorems, we get

$$\left.\begin{aligned} & A_0(X) e^{-b'} + B_0(X) e^{b'} + C_0(X) e^{-X} + D_0(X) e^{X} = 0 \\ & -\frac{X}{b'} A_0(X) e^{-b'} + \frac{X}{b'} B_0(X) e^{b'} - C_0(X) e^{-X} + D_0(X) e^{X} = 0 \\ & A_0(X) e^{b'} + B_0(X) e^{-b'} + C_0(X) e^{X} + D_0(X) e^{-X} = 0 \\ & -\frac{X}{b'} A_0(X) e^{b'} + \frac{X}{b'} B_0(X) e^{-b'} - C_0(X) e^{X} + D_0(X) e^{-X} = 0 \end{aligned}\right\} \tag{47}$$

where $b' = \sqrt{X^2 + in}$

which give

$$A_0(X) = B_0(X) = C_0(X) = D_0(X) = 0 \tag{48}$$

From coefficients of ϵ^1, we have

$$KN_1(X) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[A_1(\xi) e^{-b} + B_1(\xi) e^{b} + C_1(\xi) e^{-\xi} + D_1(\xi) e^{\xi} \right] \sin(\xi X)\, d\xi,$$

$$0 = -\sqrt{\frac{2}{\pi}} \int_0^\infty \left[-\frac{\xi}{b} A_1(\xi) e^{-b} + \frac{\xi}{b} B_1(\xi) e^{b} - C_1(\xi) e^{-\xi} + D_1(\xi) e^{\xi} \right] \cdot \cos(\xi X)\, d\xi,$$

$$-KN_2(X) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[A_1(\xi) e^{b} + B_1(\xi) e^{-b} + C_1(\xi) e^{\xi} + D_1(\xi) e^{-\xi} \right] \sin(\xi X)\, d\xi,$$

$$0 = -\sqrt{\frac{2}{\pi}} \int_0^\infty \left[-\frac{\xi}{b} A_1(\xi) e^{b} + \frac{\xi}{b} B_1(\xi) e^{-b} - C_1(\xi) e^{\xi} + D_1(\xi) e^{-\xi} \right] \cdot \cos(\xi X)\, d\xi.$$

Inverting the above four equations by Fourier sine and cosine integral theorems, we get

$$\left.\begin{aligned}
&A_1(X)\, e^{-b'} + B_1(X)\, e^{b'} + C_1(X)\, e^{-X} + D_1(X)\, e^{X} = K\bar{N}_1(X), \\
&-\frac{X}{b'} A_1(X)\, e^{-b'} + \frac{X}{b'} B_1(X)\, e^{b'} - C_1(X)\, e^{-X} + D_1(X)\, e^{X} = 0 \\
&A_1(X)\, e^{b'} + B_1(X)\, e^{-b'} + C_1(X)\, e^{X} + D_1(X)\, e^{-X} = -\, K\bar{N}_2(X), \\
&-\frac{X}{b'} A_1(X)\, e^{b'} + \frac{X}{b'} B_1(X)\, e^{-b'} - C_1(X)\, e^{X} + D_1(X)\, e^{-X} = 0
\end{aligned}\right\} \quad (49)$$

which gievs

$$\left.\begin{aligned}
A_1(X) + B_1(X) &= \frac{\frac{1}{2}\, Kb' \left[\bar{N}_1(X) - \bar{N}_2(X)\right] \sinh X}{b' \cosh b' \sinh X - X \sinh b' \cosh X}, \\
C_1(X) + D_1(X) &= \frac{\frac{1}{2}\, KX \left[\bar{N}_1(X) + \bar{N}_2(X)\right] \sinh b'}{b' \cosh b' \sinh X - X \sinh b' \cosh X}, \\
A_1(X) - B_1(X) &= \frac{\frac{1}{2}\, Kb' \left[\bar{N}_1(X) - \bar{N}_2(X)\right] \cosh X}{b' \sinh b' \cosh X - X \cosh b' \sinh X}, \\
C_1(X) - D_1(X) &= \frac{\frac{1}{2}\, KX \left[\bar{N}_1(X) + \bar{N}_2(X)\right] \cosh b'}{b' \sinh b' \cosh X - X \cosh b' \sinh X},
\end{aligned}\right\} \quad (50)$$

Making use of (48) and (50) in (39) to (41) and collecting only the real part, the complete expressions for the velocities and pressure are :

$$U(X, Z, T) = U_0(Z, T) + U'(X, Z, T) = \frac{K}{2}(1 - Z^2) \cos nT + \frac{K}{2}\sqrt{\frac{2}{\pi}}\, \epsilon$$

$$\int_0^\infty \Big[\frac{\sinh \xi \cosh \xi Z - \xi \cosh \xi \cosh \xi Z + \xi Z \sinh \xi \sinh \xi Z}{\sinh \xi \cosh \xi - \xi} \left\{ \bar{N}_1(\xi) - N_2(\xi) \right\} +$$

$$+ \frac{\xi Z \cosh \xi \cosh \xi Z + \cosh \xi \sinh \xi Z - \xi \sinh \xi \sinh \xi Z}{\sinh \xi \cosh \xi + \xi} \left\{ \bar{N}_1(\xi) + \bar{N}_2(\xi) \right\} \Big] .$$

$$\sin(\xi X) \cos nT \, d\xi \quad (51)$$

$$W(X, Z, T) = W'(X, Z, T) = -\frac{K}{2}\sqrt{\frac{2}{\pi}}\ \epsilon .$$

$$\cdot \int_0^\infty \left[\frac{\xi Z \sinh \xi \cosh \xi Z - \xi \cosh \xi \sinh \xi Z}{\sinh \xi \cosh \xi - \xi} \left\{ \bar{N}_1(\xi) - \bar{N}_2(\xi) \right\} + \right.$$

$$\left. + \frac{\xi Z \sinh \xi Z \cosh \xi - \xi \sinh \xi \cosh \xi Z}{\sinh \xi \cosh \xi + \xi} \left\{ \bar{N}_1(\xi) + \bar{N}_2(\xi) \right\} \right]$$

$$\cdot \cos(\xi X) \cos nT \, d\xi \tag{52}$$

and

$$P(X, Z, T) = P_0(X, T) + P'(X, Z, T) = C - KX \cos nT -$$

$$- K\sqrt{\frac{2}{\pi}}\ \epsilon \int_0^\infty \left[\frac{\xi \sinh \xi \cosh \xi Z}{\sinh \xi \cosh \xi - \xi} \left\{ \bar{N}_1(\xi) - \bar{N}_2(\xi) \right\} + \right.$$

$$\left. + \frac{\xi \cosh \xi \sinh \xi Z}{\sinh \xi \cosh \xi + \xi} \left\{ \bar{N}_1(\xi) + \bar{N}_2(\xi) \right\} \right] \cos(\xi X) \cos nT \, d\xi . \tag{53}$$

where C denotes a constant.

PARTICULAR CASES (SINUSOIDAL ROUGHNESS)

Case (i)

If $$N_2(X) = - N_1(X) = \sin \frac{X}{l} , \tag{54}$$

i.e., when the phase difference in the roughness of the walls is π. Here $2\pi l$ is the wavelength of roughness waves at the walls.

We may formally write

$$\bar{N}_2(\xi) = - \bar{N}_1(\xi) = \sqrt{\frac{\pi}{2}}\ \delta \left(\xi - \frac{1}{l} \right) , \tag{55}$$

where $\bar{N}_2(\xi)$ and $\bar{N}_1(\xi)$ are the Fourier sine transforms of $N_2(X)$ and $N_1(X)$ respectively and δ is the Dirac delta function.

Substituting (55) in (51) to (53) and making use of a property of Dirac delta function[9], we have

$$U(X,Z,T) = \frac{K}{2}(1-Z^2)\cos nT +$$

$$+ K\epsilon \frac{\sinh\frac{1}{l}\cosh\frac{Z}{l} - \frac{1}{l}\cosh\frac{1}{l}\cosh\frac{Z}{l} + \frac{Z}{l}\sinh\frac{1}{l}\sinh\frac{Z}{l}}{\sinh\frac{1}{l}\cosh\frac{1}{l} - \frac{1}{l}} .$$

$$. \sin\left(\frac{X}{l}\right)\cos nT, \tag{56}$$

$$W(X,Z,T) = K\epsilon \frac{\frac{1}{l}\cosh\frac{1}{l}\sinh\frac{Z}{l} - \frac{Z}{l}\sinh\frac{1}{l}\cosh\frac{Z}{l}}{\sinh\frac{1}{l}\cosh\frac{1}{l} - \frac{1}{l}} \cos\left(\frac{X}{l}\right)\cos nT \tag{57}$$

$$P(X,Z,T) = C - KX\cos nT - K\epsilon \frac{\frac{2}{l}\sinh\frac{1}{l}\cosh\frac{Z}{l}}{\sinh\frac{1}{l}\cosh\frac{1}{l} - \frac{1}{l}} \cos\left(\frac{X}{l}\right)\cos nT \tag{58}$$

Case (ii)

If $$N_2(X) = N_1(X) = \sin\frac{X}{l}, \tag{59}$$

i.e., when the phase difference in the roughness of the two walls is zero.

The expressions for U, W and P are

$$U(X,Z,T) = \frac{K}{2}(1-Z^2)\cos nT$$

$$+ K\epsilon \frac{\frac{Z}{l}\cosh\frac{1}{l}\cosh\frac{Z}{l} + \cosh\frac{1}{l}\sinh\frac{Z}{l} - \frac{1}{l}\sinh\frac{1}{l}\sinh\frac{Z}{l}}{\sinh\frac{1}{l}\cosh\frac{1}{l} + \frac{1}{l}} .$$

$$. \sin\left(\frac{X}{l}\right)\cos nT \tag{60}$$

$$W(X,Z,T) = K\epsilon \frac{\frac{1}{l}\sinh\frac{1}{l}\cosh\frac{Z}{l} - \frac{Z}{l}\sinh\frac{Z}{l}\cosh\frac{1}{l}}{\sinh\frac{1}{l}\cosh\frac{1}{l} + \frac{1}{l}} .$$

$$, \cos\left(\frac{X}{l}\right)\cos nT \tag{61}$$

$$P(X, Z, T) = C - KX \cos nT - K\epsilon \frac{\frac{2}{l} \cosh \frac{1}{l} \sinh \frac{Z}{l}}{\sinh \frac{1}{l} \cosh \frac{1}{l} + \frac{1}{l}} \cdot \cos\left(\frac{X}{l}\right) \cos nT \quad (62)$$

NUMERICAL DISCUSSION

The longitudinal and transverse velocity profiles for particular values of $\epsilon = 0 \cdot 1$ and $l = 1$ have been drawn in Fig. 1 to 5, at different cross-sections of the channel and for various values of nT. When the phase difference of the sinusoidal roughness of the plates is π, the longitudinal velocity decreases as the width of the channel increases and *vice versa*. The transverse velocity profiles in this case have a point of inflexion on the mid plane. Thus the resulting effect of these two velocities is that the direction of the flow is towards the walls if the width of the channel increases and away from the walls if the width of the channel decreases. In the other case when the phase difference of the sinusoidal roughness of the plates is zero, the longitudinal velocity is more in that portion of the channel where the wall of the channel is nearer to the x-axis and *vice versa*.

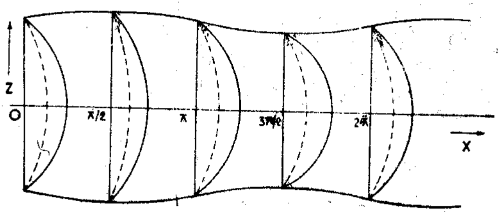

Fig. 1—The longitudinal velocity profiles at different sections of a roughness wave for $\epsilon = 0 \cdot 1$ and $nT = 0$ (——) and $nT = \pi/3$. (. . . .).

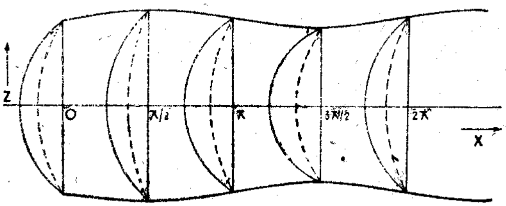

Fig. 2—The longitudinal velocity profiles at different sections of a roughness wave for $\epsilon = 0 \cdot 1$ and $nT = 2\pi/3$ (. . . .) and $nT = \pi$ (——).

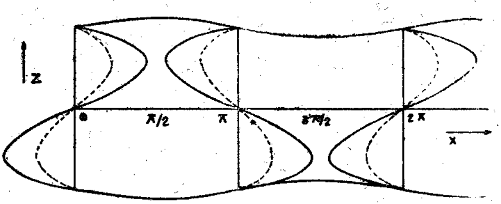

Fig. 3—The transverse velocity profiles at different sections of a roughness wave for $\epsilon = 0 \cdot 1$ and $nT = 0$ (——) and $nT = \pi/3$ (. . . .).

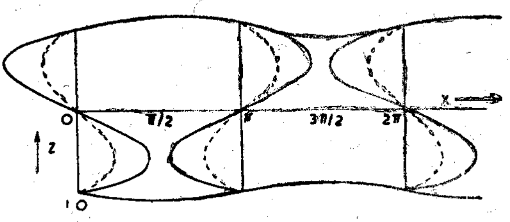

Fig. 4—The transverse velocity profiles at different sections of a roughness wave for $\epsilon = 0 \cdot 1$ and $nT = 2\pi/3$ (. . . .) and $nT = \pi$ (——).

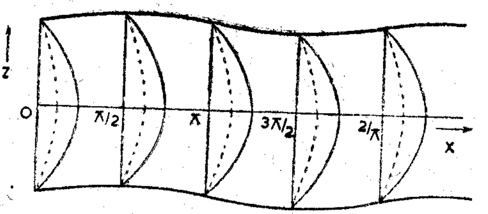

Fig. 5—The longitudinal velocity profiles at different sections of a roughness wave for $\epsilon = 0 \cdot 1$ and $nT = 0$ (——) and $nT = \pi/3$ (. . . .).

ACKNOWLEDGEMENTS

The author is indebted to Dr. P.D. Verma for his kind guidance during the preparation of this paper and he is also grateful to Prof. P.L. Bhatnagar for his constant encouragement.

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